

## Mathematical Abstraction of Year 9 Students Using Realistic Mathematics Education Based on the van Hiele Levels of Geometry

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**Abstract.** Previous research regarding abstraction has not discussed abstraction qualitatively based on van Hiele levels. Thus, it is necessary to study students' abstraction analysis based on van Hiele levels through Realistic Mathematics Education (RME) approach. The purpose of this research was to analyze mathematical abstraction based on van Hiele levels of geometry (VHLG) through RME and traditional learning approach reviewed from the levels of prior knowledge. This research employed a descriptive qualitative method involving Year 9 junior high school students as the subjects. The instruments were a mathematical abstraction test, van Hiele geometry test, and interview guidelines. The results of the high- and medium-ability students in the classroom using RME approach showed that VHLG was at the Deduction level and the abstraction ability was dominated by Empirical and Reflective Abstraction, whereas the low-ability students are at the level of Abstraction, they had imperfect Empirical and Reflective Abstraction. As for the high-ability students in the traditional learning classroom, the VHLG was at the level of Abstraction; their Reflective Abstraction was at the Representation level. While concerning the low- and medium-ability students, the VHLG was at the Analysis level; they mastered the Reflective Abstraction at the level of Recognition. This study indicates that the RME approach can trigger the development of mathematical abstraction, and accelerate the van Hiele levels of geometry progress.

**Keywords:** geometry, mathematical abstraction, van Hiele levels

### Introduction

Mathematical abstraction is one of the skills students must acquire as it is the path for the emergence of mathematical concepts. Mathematical concepts should be constructed in the students' minds through meaningful learning processes, not transferred directly, nor emphasizing students to memorize it only (Fitriani, Suryadi, & Darhim, 2018b). The concept construction process in students' minds by utilizing their initial experience or knowledge is called mathematical abstraction process (Nurhasanah, Kusumah, & Sabandar, 2017). Such a process occurs when a person realizes the same characteristics in an object based on his/her experiences. These similarities are used as a basis for classification (Skemp, 2012). That way, one can recognize a new experience by comparing it with the ones that have been previously formed in mind. The results of the abstraction process are concepts.

Some researchers have reviewed the issue (Dewi, Siregar, & Andriani, 2018; Drager & Hansen, 2014; Ferrari, 2003; Komala, 2018; Mitchelmore & White, 2007; Skemp, 2012; Subroto & Suryadi, 2018; White & Mitchelmore, 2010). However, generally, previous studies focused on topics outside of geometry. Mitchelmore & White (2007) argued that the process

of abstraction has a vital role in geometry learning. Through the process of abstraction, students identify an object by observing the commonality, classifying the characteristics, discovering the properties of a concept, and constructing a concept of each object.

Based on a cognitive point of view, abstraction is one of the reasons for failure in mathematics (Ferrari, 2003). The failure is thought to be related to the way of forming mathematical concepts. As stated earlier, the concept cannot be done by directly delivering information only; instead, it requires students to experience the process directly. However, nowadays learning mainly occurs deductively, avoiding empirical induction. Therefore, it prevents the abstraction process and ultimately makes it more difficult for students. This is in line with the results of a preliminary study reported that junior high school students generally find it challenging to understand geometry and they are unable to carry out abstraction processes (Fitriani, Suryadi, & Darhim, 2018a). Learning difficulties in geometry were alleged because students were unable to conduct an abstraction process to understand the concepts. This becomes a severe problem because the abstraction process cannot be separated from learning mathematics. Therefore, it is vital to analyze further the abstraction process occurred in geometry.

According to Mason theory of mental development (Mason, 1998), van Hiele geometric level of thinking consisted of five stages. First, Visualization stage (level 0), where abstraction abilities are not yet measurable. Second, the Analysis stage (level 1), students can express their thinking in the form of mathematical symbols, words or diagrams. Third, the Abstraction stage (level 2), the deductive thinking of students have started to grow. Concerning the process involving abstraction, students can reorganize and make new mathematical elements so that they can connect the previous concept with a new one. Fourth, Deduction stage (level 3) where students can withdraw deductive conclusions, i.e., concluding a general to a unique form. The process of abstraction reveals that students can provide reasoning for the decisions in generalizing and showing a summary of their activities. Fifth, accuracy stage (level 4), students understand the importance of the accuracy of the fundamentals underlying proof.

The previous description suggests a connection between the abstraction process and the van Hiele thinking level. Each level reflects the aspects of abstraction indicating the students' abstraction ability in learning geometry develops following the van Hiele thinking level. Although van Hiele does not directly mention the interrelationships between abstractions in the learning stages of geometry, yet within each level, there is considerable abstraction involvement. Thus, the abstraction has an essential role in the process of learning geometry.

Students' thinking level can move to higher stages based on designed learning, one of which is the RME approach. RME has a unique characteristic, that is a self-developed model (Gravemeijer, 2015). This characteristic greatly facilitates the occurrence of mathematical abstraction processes. The self-developed model aims to bridge the gap between informal and formal mathematics knowledge (Johar, 2013). The use of the model connects the concrete to the abstract level of mathematics. The model furthers the process of abstraction (Endramawati, Mastur, & Mariani, 2019).

Piaget distinguishes abstraction into three kinds, i.e., Empirical, Pseudo-Empirical, and Reflective. Reflective abstraction is the general coordination of action so that the source is a subject equipped with complete internal properties. In this case, it focuses on the idea of action and operation being a thematic object to thinking or arguably with assimilation. The result of a reflective abstraction is a knowledge scheme.

The abstraction process is an activity when we realize the characteristics are similar to previous experiences (Skemp, 2012). Then, these similarities are used to classify so that we can recognize a new experience by comparing it with the previous ones formed in our minds. To distinguish between abstracting as an activity and abstraction as a final product, we will then refer to the final product as a concept. Based on these descriptions, the abstraction elaborated by Skemp belongs to empirical abstraction as it is based on experience.

Abstraction is divided into empirical and theoretical (Mitchelmore & White, 2007). The flow of empirical and theoretical abstraction processes are different. In empirical, individuals form new concepts based on observation and experience. In theoretical, matching old concepts with experiences in individual thinking will develop new concepts. Piaget's theory of reflective abstraction is a form of theoretical abstraction (Tall, 1991). Based on the previous description, the mathematical abstraction process in this study is the construction of concepts that occur in students' mind. The construction takes advantage of the student's initial experience or knowledge. Thus, a mathematical abstraction process is Empirical. The reflective level is recognition, representation, structural abstraction and structural awareness (Cifarelli, 1988).

Based on the studies previously described, it is clear that limited research has discussed qualitatively abstraction based on van Hiele level. Thus, it is necessary to study the students' abstraction analysis based on van Hiele level using the RME approach.

There are two research questions in this study: 1) how does students' mathematical abstraction based on van Hiele geometry thinking level using RME in the learning process?; 2) how does students' mathematical abstraction based on van Hiele geometry thinking level in traditional learning?

## Method

This research employed a descriptive qualitative method. The participants were Year 9 students from two classes in one of junior high school in Ngamprah, West Java, Indonesia. A pre-test, a prerequisite test, was administered to students in both classes at the beginning of the study to examine their initial ability. The pre-test results categorized the students into three groups: low-, medium-, and high-ability students. One student was then selected from each group in the two classes based on teachers' consideration. The students were selected based on their high communicative skills, so it would be easier to explore the information further. Learning took place for two months in both classes. At the end of the teaching period, these two groups of students again took van Hiele geometry and abstraction tests. Interviews were then conducted on selected students for triangulation purpose of the data.

The instruments in this research were a mathematical abstraction and van Hiele geometry tests. The indicators of mathematical abstraction ability consisted of empirical abstraction, theoretical abstraction, and reflective abstraction were the Indicators of used. Reflective Abstraction consists of several levels: recognition, representation, structural abstraction, and structural awareness (Cifarelli, 1988).

Data were analyzed in depth based on each written answer provided by the participants. The analysis results then determined students' abstraction ability which was then associated with van Hiele's levels. The students who fulfilled all indicators of abstraction skills were expected to experience fast progress in van Hiele's levels.

## Results and Discussion

Table 1 presents a recapitulation of mathematical abstraction test results and VHGT for the selected topics. Table 1 shows the progress of level before and after the treatment. S1, S2, and S3 were the high, medium, and low ability students from RME class respectively, while S1, S2, S3 were the high, medium, and low ability students from traditional class respectively. Before treatment, students had already studied geometry (the concept of plane geometry and polyhedron) with traditional learning. Their geometry level was at level 1 (visualization). They tended to recognize a form of geometry by paying attention to plane geometry and polyhedron visually without knowing its properties. This case occurs because geometry is considered as the subject of memorizing and calculation. Students must memorize definitions as well as arguments without knowing the process of discovering the concepts. Students also are unable to apply it in everyday life situation nor transfer it into a new context. Such learning does not promote the development of students' abstraction skills or the van Hiele level; it will only remain at level 1, namely visualization.

Table 1. Recapitulation of mathematical abstraction test results and van Hiele geometry test

Approach	Subject	Abstraction Ability	Category of van Hiele Level	
			Before Treatment	After Treatment
RME	S1	Empirical Abstraction & Reflective Abstraction	1	4
	S2	Empirical Abstraction & Reflective Abstraction	1	4
	S3	Empirical Abstraction & Reflective Abstraction (Structural Abstraction)	1	3
Traditional	S4	Reflective Abstraction (Representation)	1	3
	S5	Reflective Abstraction (Recognition)	1	2
	S6	Reflective Abstraction (Recognition)	1	2

The progress of level between the RME and the traditional approach classroom was different. For RME classroom, students tended to have excellent abstraction skills, as well as faster progress in reaching the next level of van Hiele’s stages. As an example, the following will show some student’s work in solving the problem regarding the abstraction abilities in learning mathematics. In the RME approach, students were directed to the abstraction process which in turn indirectly encouraged the transition process of the van Hiele level to a higher direction. In the RME approach classroom, students were directed to do horizontal mathematical toward vertical mathematics; it helped students in carrying out the abstraction process.

Below is the first problem that was given to examine the students’ abstraction and van Hiele level skills.

**The first problem**

Given the  $DC$  length equal to  $AB$  is 88 cm, Angle  $A = B = C = D = 90^\circ$ , then radius of the circle is 14 cm. With a known size, do you think that the image can form a tube? Try to analyze and conclude!



Here are the results of S1 and S6,

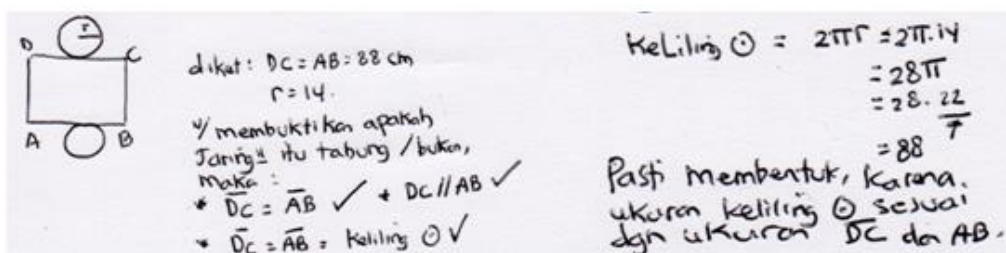
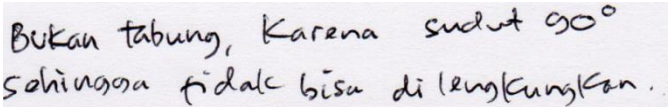


Figure 1. Abstraction test results from S1

Based on the results of the work of S1 presented in Figure 1, S1 seems to be able to recall the experience of previous concepts. For example, the idea of parallelism, segment length, and circumference of circles supporting the given problem. The student could detail in his/her own words to express what was needed to form a cylinder from a given image. Based on his/her experience with such concepts, he/she could reorganize his knowledge into a whole idea, and conclude that the image could form a cylinder. The student's empirical abstraction and reflective abstraction abilities were good enough, as supported by the reasons (rules) he/she showed in his/her activities (Cifarelli, 1988). Students recognize various forms of line representation such as trajectories of paper folds, line drawings, and others. Students will know the same characteristics by experiencing real objects.

Based on the results of VHGT and associated with the work, this student could understand and used the theorem, at the level of Deduction (Mason, 1998). The students can make deductive conclusions, i.e., concluding the general to a unique form. The abstraction process reveals that students can provide reasoning for the decisions in generalizing and show a summary of their activities. The abstraction process will be realized based on someone's experiences. This concept is in line with the philosophy of empiricism and the theory of a posteriori stating that all knowledge originates from experience. The following is the student interview transcript,

- R : Why do you think that the image is a cylinder?*  
*S1 : Because DC and AB are the same lengths, so the upper and the base size is congruent.*  
*R : Are DC and AB parallel?*  
*S1 : Yes, because four angles are formed by  $90^\circ$ , so according to the theorem the line segments facing it must be parallel.*  
*R : Can the circle with the radius of 14 cm cover the cylinder?*  
*S1 : Of course, the circumference of the circle is 88 cm, and the size is equal to the length of AB and DC, so it can surely cover with the cylinder.*



Bukan tabung, karena sudut  $90^\circ$  sehingga tidak bisa dilengkungkan.

Figure 2. Abstraction test results from S6

Based on the results of S6 presented in figure 2, students were unable to recall previous experiences to support the problem, so that he/she was unsuccessful in solving the problem. Based on the VHGT results, S6 was at the level of Analysis, he/she already recognized the nature of geometry, could make observations and measurements, but mistakenly interpreted the problem. This finding is in line with Mason (1998), in this level, students are familiar with the properties of geometry and able to name its elements. At this level, the abstraction abilities

involve recalling previous activities and experiences related to the problem. The following is the student interview transcript,

- R : Why did you say that it was not a cylinder?  
 S6 : Because the angle is  $90^\circ$ , so it is not a cylinder.  
 R : Why is that?  
 S6 : The cylinder is a curved-face three-dimensional object if the angle is  $90^\circ$  it cannot be wrapped, hmmm ... however, which part is it? (the student asking himself then silent).

Below is the second problem that given to examine the students' abstraction and van Hiele level skills.

**The Second Problem**

A cup has a circular top and base with a radius of 18 cm and 6 cm respectively. The distance between the top of the cup to the bottom is 20 cm. Can you draw it? How it will look like? Which concept do you use to determine the maximum water volume fit in it? Can you determine the amount of water!

Here are the result of the S12,

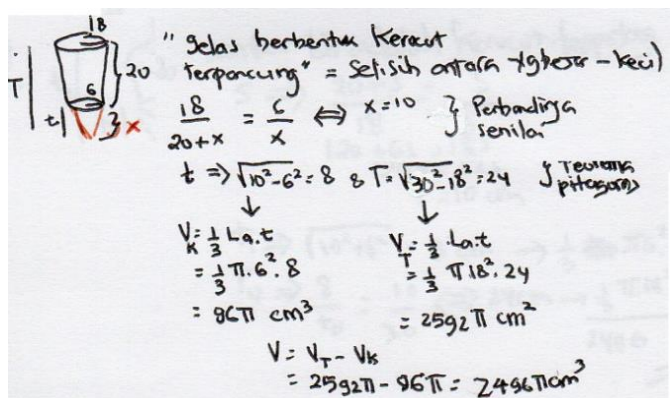


Figure 3. Abstraction test results of S2

Figure 3 is the work of S2. The student had an experience with previous concepts supporting the given problem, such as the concept of direct proportion and Pythagoras. He could create a picture of a cup and suspected that it was part of a cut off cone. Based on the idea, the problem can be solved. From the assumption that the cup is a cone, the student organizes (collects, compiles, develops) known elements such as the radius and length with the concept of direct proportion and Pythagoras so that the cup volume can be determined. Here, mathematics in context introduces the concepts in realistic contexts that support mathematical abstraction (Endramawati, Mastur, & Mariani, 2019). The rules that he/she conveyed in his/her activities are good enough so that the students' reflective abstraction abilities are complete at all levels (Cifarelli, 1988). Judging from the results of VHGT and the associated work, the student could

understand and use the theorem, and he was at the level of Deduction (Mason, 1998). We can see that van Hiele's level of thinking does not depend on age or maturity of an individual, but the moving pace from one to the next level relies much more on the learning experience (Mason, 1998). That is why teachers have to provide a learning experience fitting the students' level of thinking; one of them is through the RME approach. The following is the student interview transcript,

- R : When reading the matter, what's on your mind?  
 S2 : After looking at the picture of the cup, I immediately imagine it is a truncated cone.  
 R : Why do you think so?  
 S2 : At first I sketched in full, then I was sure that it was a truncated cone. Then, I thought that for the essential unknown figures I could look for with the concept of proportion, I try first with other numbers, it turned out matching with the direct proportion  
 R : Then?  
 S2 : I found the volume of the whole cone and the small cone, then looking for the difference between the two.

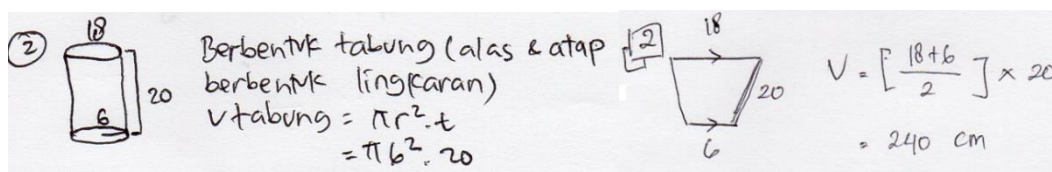


Figure 4. Abstraction test results from S5 and S6

Based on the results of S5 and S6 presented in Figure 4, it appears that students could recall previous experiences related to the given problem. S5 suspected that the question concerned the concept of the cylinder. The pieces of information captured were that the circular bases (top and bottom), so the formula should be  $\pi r^2 t$ . S5 did not notice that the radius of the top and bottom bases were different. S6 committed a further mistake by sketching 2-dimensional object looking like a trapezoid. Both S5 and S6 already recognized the nature of a geometry shape and could mention the elements; however there was a slight misconception resulting in inappropriate conclusion. The following is the student interview transcript,

- R : Try to explain the sketch you have made!  
 S5 : I sketched a cylinder.  
 R : Why? Does the cylinder have a congruent size of top and bottom bases?  
 S5 : I don't know (looks pensive), but the top and bottom bases are circular.  
 S6 : If we look it sideways, the cup is trapezoidal, so it is easy to determine its volume, just by summing up the radius, then divided by two and finally multiplied by the height.



Based on the results of the interview, S5 and S6 still had a lack of understanding of the problems. They had a lack of experiences related to the concept. So, it could not be a basis for recognizing a new experience. For low-ability students, it is challenging to achieve abstraction, because their prior knowledge is limited (Fitriani, Suryadi, & Darhim, 2018a). Students with limited prior knowledge indicating inadequate prerequisite knowledge regarding the concepts, they will find it difficult to construct new knowledge. The connection bridging the concepts does not occur so that the formation of new knowledge in the abstraction process becomes impossible. Furthermore, the van Hiele level will not advance, because basically, the transition to the higher direction of van Hiele level requires adequate basic knowledge as it will continue to be associated with the knowledge built.

The results of this study indicate that the abstraction process development of students learning using RME approach was better than those learning using the traditional approach. This finding is due to RME emphasizing the self-developed model that facilitate students to develop their informal to formal knowledge gradually (Gravemeijer, 1994; Johar, Patahuddin, & Widjaja, 2017).

Abstraction consists of new concepts and schemes; both elements accelerate or support the formation of new knowledge and strengthen the existing knowledge (Skemp, 2012). This factor promotes the acceleration of progressing to a higher level in van Hiele's thinking level. Besides, the van Hiele geometry learning process prioritizes the existing knowledge and active interactions. Interaction is one of the characteristics of RME that can improve the students thinking skills (Widjaja, Dolk & Fauzan, 2010).

## **Conclusion**

The results reveal that there is a progress of thinking level on each subject regarding the van Hiele thinking level. For S1 and S2, van Hiele's thinking level after the learning is at the Deduction level. They have excellent empirical and reflective abstraction. S3 is at the level of abstraction with imperfect Empirical Abstraction and Reflective Abstraction not in all aspects (only up to structural abstraction level). For S4, the geometric thinking level is at the abstraction level and a Reflective Abstraction only (which is also incomplete, only at the representation level). Finally, S5 and S6 are at the analysis level and only master the ability of reflective abstraction at the recognition level.

The results indicate that the RME approach applied can lead to the development of students' abstraction abilities and accelerate the transfer of students' van Hiele thinking levels. The abstraction in term of new concepts and schemes, these two elements accelerate or support the formation of new knowledge and strengthen the existing knowledge which in turn

supporting the acceleration of advancing to a higher level in van Hiele's thinking. Besides, the geometry learning process following van Hiele prioritizes existing knowledge and active interactions. These interactions are supported by existing knowledge and are a learning activity packed in the abstraction process.

The abstraction process was not well-developed in traditional learning where high-ability students could only achieve reflective abstraction at the level of representation. This happened because the teaching did not encourage students in forming the concept. Instead, they tend to be provided with a finished concept and encourage to memorize. Thus the van Hiele's level progress was slow.

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