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Flexural Free Vibration of A Straight Vertical Cantilever Beam

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Abstract
In this paper, flexural free vibration of a straight vertical cantilever beam have been modeled and simulated. Here, a modeled-cantilever beam has modulus of elasticity, moment of inertia, cross-section and density are constant. Motion equation of a modeled-cantilever beam are separated become two Partial Differential Equations; one depends on position and another within time. This technique yields the motion equation of a modeled-cantilever beam contains two functions; one defines deflection shapes and another defines amplitude of vibration within time. The deflection shapes of a modeled-cantilever beam are described in first five natural frequencies. Furthermore, the motion equation of a modeled-cantilever beam is solved by using Fourier series. From simulation of a modeled-cantilever beam with 2 GPa modulus of elasticity, 2.67x10^{-8} m^4 moment of inertia, 8x10^{-4} m^2 cross-section, 7862.30 kg/m^3 density, 1 m length, and 100 N initial load obtained first five natural frequencies respectively 16.29, 102.11, 285.95, 560.36, and 926.22 rad/s.

Keywords: flexural free vibrations, a straight vertical cantilever beam, Partial Differential Equations, deflection shapes, amplitude of vibrations, Fourier series, natural frequencies

1. Introduction
Beam-type structures are widely used in many branches of modern aerospace, mechanical and civil engineering [1]. In mechanical engineering, there are numerous studies dealing with problems related to beam. There are different types of beam models. One of the well-known models is the Euler-Bernoulli beam theory that works well for cantilever beams [2-11]. The Partial Differential Equation and Fourier series will be utilized for the solution of cantilever beam with initial loading in order to arrive at a free vibration. One of the methods of solving the equation of the free vibration is the separation of the variables which assumes that the solution is the product of two functions, one defines the deflection shape and the other defines the amplitude of vibration within time [12-19].

Objectives of this work are derive of Partial Differential Equation (PDE) of flexural free vibration of cantilever beam, determine mode shapes and natural frequencies of cantilever beam, and simulate free vibration of a straight vertical flexural of cantilever beam with initial loading. Here is used two physical assumptions, i.e. cantilever beam will vibrate at its characteristic frequencies as fourth-order free vibration of undamped system; and the elastic modulus, moment of inertia, cross sectional area, and density are constant along the beam length [13].

2. Material and Methods
A straight vertical cantilever beam under a horizontal load will deform into a curve. Figure 1 shows such a beam. One end of the beam is fixed, while the other end is free. When this force is removed, the beam will return to its original shape. However, its inertia will keep the beam in motion. Thus, the beam will vibrate at its characteristic frequencies.

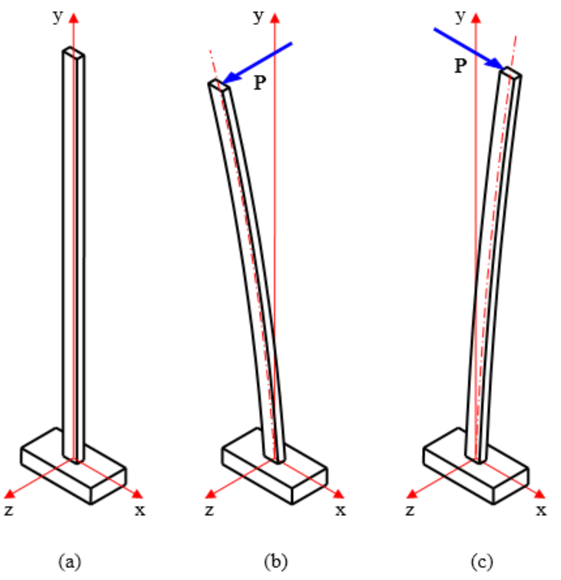

Figure 1. The free end of a cantilever beam is subjected to a point load: (a) No load applied to cantilever beam, (b) z-axis load, P, applied to cantilever beam and (c) x-axis load, P, applied to cantilever beam.

See Figure 1(b) and 1(c) above, if the free end of a cantilever beam is subjected to a point load, P, the beam will deflect into a curve. The larger the load applied to beam, the greater the deflection of beam.

Assuming the beam undergoes small deflections, is in the linearly elastic region, and has a uniform cross-section and properties; the following equations can be used. If a point load applied to cantilever beam as shown by Figure 2, the curvature of the beam K is equal to the second derivative of the deflection.

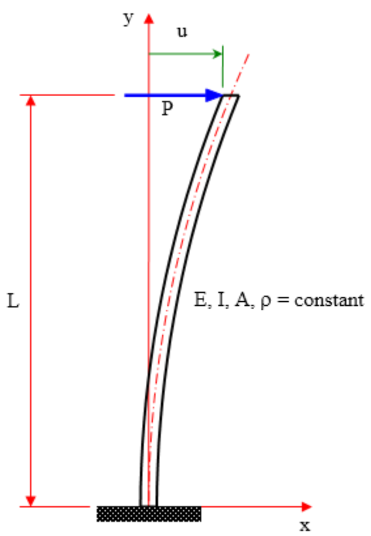

Figure 2. A point load, P, applied to cantilever beam.

$$K = \frac{\partial^2 u}{\partial y^2} \tag{1}$$

The curvature can also be related to the bending moment, M, and the flexural rigidity, EI,

$$K = -\frac{M}{EI} \tag{2}$$

where E is the elastic modulus of the beam and I is the moment of inertia. The bending moment in a beam can be related to the shear force, Q, and the lateral load, N, on the beam. Thus,

$$M = -EI\frac{\partial^2 u}{\partial y^2} \tag{3}$$

$$Q = EI\frac{\partial^3 u}{\partial y^3} \tag{4}$$

$$N = EI\frac{\partial^4 u}{\partial y^4} \tag{5}$$

For the load shown in Figure 2, the distributed lateral load, shear force, and bending moment respectively are

$$N(y) = 0 \tag{6}$$

$$Q(y) = P \tag{7}$$

$$M(y) = -PL\left(1 - \frac{y}{L}\right) \tag{8}$$

Thus, the solution of u(y) at Equation (3) is

$$\frac{\partial u}{\partial y} = -\frac{1}{EI}\int_{y=0}^{y} M(y)dy = \frac{PL}{EI}\left(y - \frac{y^2}{2L}\right) \tag{9}$$

$$u(y) = \int_{y=0}^{y} \frac{\partial u}{\partial y}dy = \frac{PL}{EI}\left(\frac{y^2}{2} - \frac{y^3}{6L}\right) \tag{10}$$

At the free end of the beam, the displacement is

$$u(L) = \frac{PL^3}{3EI} \tag{11}$$

A. Derivation of the Partial Differential Equation of the Cantilever Beam

When the force, P, is removed from a displaced beam, the beam will return to its original shape. However, inertia of the beam will cause the beam to vibrate around that initial location. Assuming the elastic modulus, inertia, cross sectional area (A), and density (ρ) are constant along the beam length. To derive equation for that vibration, a dy length of element at one point lies at y from fixed end would be analyzed as shown Figure 3.

The equilibrium of x-axis forces is

$$-(Q)+\left(Q+\frac{\partial Q}{\partial y}dy\right)-\left(\rho A dy\frac{\partial^2 u}{\partial t^2}\right)=0 \tag{12}$$

and the equilibrium of moment is

$$-(M)-(Qdy)+\left(M+\frac{\partial M}{\partial y}dy\right)=0 \tag{13}$$

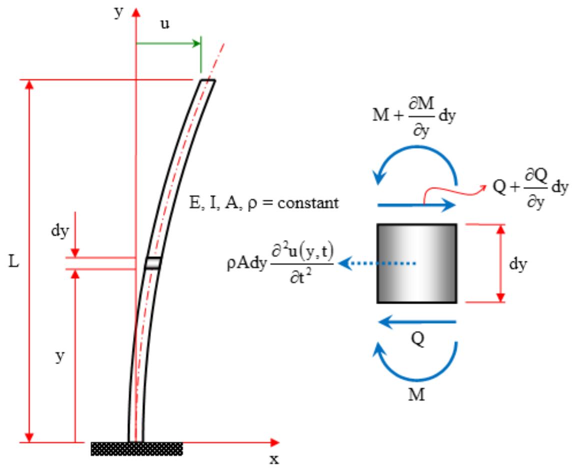

Figure 3. Cantilever beam modeling to derive vibration equation.

With rearranging equation (12), obtained

$$\frac{\partial Q}{\partial y}dy=\rho A dy\frac{\partial^2 u(y,t)}{\partial t^2} \tag{14}$$

and with rearranging equation (13), obtained

$$Q=\frac{\partial M}{\partial y} \tag{15}$$

Substituting equation (15) into equation (14), gives

$$\frac{\partial}{\partial y}\left(\frac{\partial M}{\partial y}\right)dy=\rho A dy\frac{\partial^2 u(y,t)}{\partial t^2} \tag{16}$$

$$\frac{\partial^2 M}{\partial y^2}dy=\rho A dy\frac{\partial^2 u(y,t)}{\partial t^2} \tag{17}$$

Because of $M=-EI\frac{\partial^2 u(y,t)}{\partial y^2}$, then

$$\frac{\partial^2}{\partial y^2}\left(-EI\frac{\partial^2 u(y,t)}{\partial y^2}\right)dy=\rho A dy\frac{\partial^2 u(y,t)}{\partial t^2} \tag{18}$$

$$EI\frac{\partial^4 u(y,t)}{\partial y^4}=-\gamma_m\frac{\partial^2 u(y,t)}{\partial t^2} \tag{19}$$

where

$$\gamma_m=\rho A \tag{20}$$

γ_m is the linear mass density of the beam.

Equation (19) is best solved by separation of variables. Assume that the displacement can be separated into two parts; one depends on position and another on time.

$$u(y,t)=U(y)\eta(t) \tag{21}$$

where U is independent of time, and η is independent of position.

Substituting equation (21) into eqution (19), obtained

$$EI\frac{\partial^4[U(y)\eta(t)]}{\partial y^4}=-\gamma_m\frac{\partial^2[U(y)\eta(t)]}{\partial t^2} \tag{22}$$

where

$$\frac{\partial^4[U(y)\eta(t)]}{\partial y^4}=\eta(t)\frac{\partial^4 U(y)}{\partial y^4} \tag{23}$$

and

$$\frac{\partial^2[U(y)\eta(t)]}{\partial t^2}=U(y)\frac{\partial^2 \eta(t)}{\partial t^2} \tag{24}$$

Therefore,

$$EI\eta(t)\frac{\partial^4 U(y)}{\partial y^4}=-\gamma_m U(y)\frac{\partial^2 \eta(t)}{\partial t^2} \tag{25}$$

Dividing equation (25) by $\gamma_m U(y)\eta(t)$, obtained

$$\frac{EI}{\gamma_m U(y)}\frac{\partial^4 U(y)}{\partial y^4}=-\frac{1}{\eta(t)}\frac{\partial^2 \eta(t)}{\partial t^2} \tag{26}$$

Since the left side of equation (26) does not change as t varies, the right side must be a constant. Similarly, since the right half of equation (26) does not change as y varies, the left half must be a constant. Because each side is constant, equation (26) is valid and the method of separation of variables may be used.

$$\frac{EI}{\gamma_m U(y)}\frac{\partial^4 U(y)}{\partial y^4}=-\frac{1}{\eta(t)}\frac{\partial^2 \eta(t)}{\partial t^2}=\omega^2 \tag{27}$$

From equation (27), be have the first Ordinary Differential Equation (1st ODE)

$$\frac{EI}{\gamma_m U(y)}\frac{\partial^4 U(y)}{\partial y^4}=\omega^2 \tag{28}$$

and the second Ordinary Differential Equation (2nd ODE)

$$-\frac{1}{\eta(t)}\frac{\partial^2 \eta(t)}{\partial t^2}=\omega^2 \tag{29}$$

Rearranging equation (28), obtained

$$\frac{\partial^4 U(y)}{\partial y^4}=\frac{\omega^2\gamma_m U(y)}{EI}$$
$$\frac{\partial^4 U(y)}{\partial y^4}-\frac{\omega^2\gamma_m U(y)}{EI}=0 \tag{30}$$

Assume that

$$k^4=\frac{\omega^2\gamma_m}{EI}\quad ; \quad \omega^2=\frac{k^4 EI}{\gamma_m}\quad ; \quad \omega=k^2\sqrt{\frac{EI}{\rho A}} \tag{31}$$

Then,

$$\frac{\partial^4 U(y)}{\partial y^4}-k^4U(y)=0 \tag{32}$$

With rearranging equation (29), obtained

$$-\frac{\partial^2 \eta(t)}{\partial t^2}=\omega^2\eta(t)$$
$$\frac{\partial^2 \eta(t)}{\partial t^2}+\omega^2\eta(t)=0 \tag{33}$$

To solve equation (32), assume $U(y)=Ce^{\lambda t}$ and $\frac{\partial^4 U(y)}{\partial y^4}=\lambda^4 Ce^{\lambda t}$. Then, equation (32) become

$$\lambda^4 Ce^{\lambda t}-k^4 Ce^{\lambda t}=0 \tag{34}$$

From equation (34), obtained

$$\left(\lambda^4-k^4\right)Ce^{\lambda t}=0$$
$$\left(\lambda^4-k^4\right)=0$$
$$\left(\lambda^2+k^2\right)\left(\lambda^2-k\right)^2=0$$
$$\lambda^2=-k^2\quad ; \quad \lambda^2=k^2$$
$$\lambda=\pm ik\quad ; \quad \lambda=\pm k \tag{35}$$

So, the general solution is given by:

$$U(y)=C_1\cos(ky)+C_2\sin(ky)+C_3e^{ky}+C_4e^{-ky} \tag{36}$$

and

$$\frac{\partial U(y)}{\partial y}=-C_1k\sin(ky)+C_2k\cos(ky)+C_3ke^{ky}-C_4ke^{-ky} \tag{37}$$

In order to solve equation (36) and (37), the following boundary conditions for a cantilever beam are needed

$$u(0,t)=U(0)=0$$
$$\frac{\partial u(0,t)}{\partial y}=\frac{\partial U(0)}{\partial y}=0$$
$$\frac{\partial^2 u(L,t)}{\partial y^2}=\frac{\partial^2 U(L)}{\partial y^2}=0$$
$$\frac{\partial^3 u(L,t)}{\partial y^3}=\frac{\partial^3 U(L)}{\partial y^3}=0$$

Substituting boundary conditions into equation (36) and (37), so yields equation (38) and (39).

$$C_3=-\frac{1}{2}\left(C_1+C_2\right) \tag{38}$$
$$C_4=-\frac{1}{2}\left(C_1-C_2\right) \tag{39}$$

Furthermore, substitution equation (38) and (39) into equation (36), obtained

$$\begin{aligned}U(y)&=C_1\cos(ky)+C_2\sin(ky)-\frac{1}{2}\left(C_1+C_2\right)e^{ky}\\&\quad-\frac{1}{2}\left(C_1-C_2\right)e^{-ky}\\&=C_1\cos(ky)+C_2\sin(ky)-\frac{1}{2}C_1\left(e^{ky}+e^{-ky}\right)\\&\quad-\frac{1}{2}C_2\left(e^{ky}-e^{-ky}\right)\\&=C_1\cos(ky)+C_2\sin(ky)-C_1\cosh(ky)\\&\quad-C_2\sinh(ky)\end{aligned}$$

$$\begin{aligned}&=C_1\left[\cos(ky)-\cosh(ky)\right]\\&\quad+C_2\left[\sin(ky)-\sinh(ky)\right]\end{aligned} \tag{40}$$

and obtained first, second and third order derivative from equation (40).

$$\frac{\partial U(y)}{\partial y}=C_1\left[-\sin(ky)-\sinh(ky)\right]+C_2\left[\cos(ky)-\cosh(ky)\right] \tag{41}$$

$$\frac{\partial^2 U(y)}{\partial y^2}=C_1\left[-\cos(ky)-\cosh(ky)\right]+C_2\left[-\sin(ky)-\sinh(ky)\right] \tag{42}$$

$$\frac{\partial^3 U(y)}{\partial y^3} = C_1[\sin(ky) - \sinh(ky)] + C_2[-\cos(ky) - \cosh(ky)] \tag{43}$$

From the boundary condition for second and third order derivative substitution into equation (42) and (43), yields

$$\frac{\partial^2 U(L)}{\partial y^2} = C_1[-\cos(kL) - \cosh(kL)] + C_2[-\sin(kL) - \sinh(kL)] = 0$$

$$C_1[\cos(kL) + \cosh(kL)] + C_2[\sin(kL) + \sinh(kL)] = 0 \tag{44}$$

$$\frac{\partial^3 U(L)}{\partial y^3} = C_1[\sin(kL) - \sinh(kL)] + C_2[-\cos(kL) - \cosh(kL)] = 0$$

$$C_1[\sin(kL) - \sinh(kL)] + C_2[-\cos(kL) - \cosh(kL)] = 0 \tag{45}$$

Then, from equation (44), gives

$$C_2 = -C_1 \frac{\cos(kL) + \cosh(kL)}{\sin(kL) + \sinh(kL)} \tag{46}$$

Substitution equation (46) in to equation (40), obtained

$$\begin{aligned} U(y) &= C_1[\cos(ky) - \cosh(ky)] - C_1 \frac{\cos(kL) + \cosh(kL)}{\sin(kL) + \sinh(kL)}[\sin(ky) - \sinh(ky)] \\ &= C_1 \left\{ \begin{aligned} &[\cos(ky) - \cosh(ky)] \\ &- \frac{\cos(kL) + \cosh(kL)}{\sin(kL) + \sinh(kL)}[\sin(ky) - \sinh(ky)] \end{aligned} \right\} \\ &= C \left\{ \begin{aligned} &[\cos(ky) - \cosh(ky)] \\ &- \frac{\cos(kL) + \cosh(kL)}{\sin(kL) + \sinh(kL)}[\sin(ky) - \sinh(ky)] \end{aligned} \right\} \end{aligned} \tag{47}$$

Furthermore, to solve equation (33), assume $\eta(t) = Ae^{\sigma t}$ and $\frac{\partial^2 \eta(t)}{\partial t^2} = \sigma^2 ACe^{\sigma t}$. Then, equation (33) become

$$\sigma^2 Ae^{\sigma t} + \omega^2 Ae^{\sigma t} = 0 \tag{48}$$

From equation (48), obtained

$$\begin{aligned} &(\sigma^2 + \omega^2)Ae^{\sigma t} = 0 \\ &(\sigma^2 + \omega^2) = 0 \\ &\sigma^2 = -\omega^2 \end{aligned}$$

$$\sigma = \pm i\omega \tag{49}$$

So, be have the general solution is given by:

$$\eta(t) = A\cos(\omega t) + B\sin(\omega t) \tag{50}$$

Equation (47) and (50) give general solution of flexural free vibration of a straight vertical cantilever beam.

$$u(y,t) = C \left\{ \begin{aligned} &[\cos(ky) - \cosh(ky)] \\ &- \frac{\cos(kL) + \cosh(kL)}{\sin(kL) + \sinh(kL)}[\sin(ky) - \sinh(ky)] \end{aligned} \right\} \times [A\cos(\omega t) + B\sin(\omega t)] \tag{51}$$

B. Equations in Normal Coordinate

Equation (51) is free vibration equation. It can be write in general form

$$u = \Phi(y)\,[A\cos(\omega t) + B\sin(\omega t)] \tag{52}$$

where $\Phi(y)$ is mode shape equation cantilever beam

$$\Phi(y) = C \left\{ \begin{aligned} &[\cos(ky) - \cosh(ky)] \\ &- \frac{\cos(kL) + \cosh(kL)}{\sin(kL) + \sinh(kL)}[\sin(ky) - \sinh(ky)] \end{aligned} \right\} \tag{53}$$

Equation (52) can be written in form

$$u = \sum_{i=1}^{\infty} \{\Phi_i(y)\,[A_i \cos(\omega_i t) + B_i \sin(\omega_i t)]\} \tag{54}$$

From equation (19), gives

$$u^{iv} + \frac{\gamma_m}{EI}\ddot{u} = 0$$

$$\frac{d^4 \Phi(y)}{dy^4}[A\cos(\omega t) + B\sin(\omega t)] - \frac{\omega^2 \gamma_m}{EI}\Phi(y)[A\cos(\omega t) + B\sin(\omega t)] = 0 \tag{55}$$

Assume $k^4 = \frac{\omega^2 \gamma_m}{EI}$, so, equation (55) become

$$\frac{d^4\Phi(y)}{dy^4} - k^4\Phi(y) = 0 \tag{56}$$

Equation (56) can be written in form

$$\Phi_i^{iv}(y) = k^4\Phi_i(y) \tag{57}$$

and have eigenvalue λ_i where

$$\lambda_i = k_i^4 = \frac{\omega^2\gamma_m}{EI} \tag{58}$$

Thus, equation (57) can be written

$$\Phi_i^{iv}(y) = \lambda_i\Phi_i(y) \tag{59}$$

Orthogonally of equation (59) can be known if consider i-mode and j-mode of that equation. So,

$$\Phi_i^{iv}(y) = \lambda_i\Phi_i(y) \text{ and } \Phi_j^{iv}(y) = \lambda_j\Phi_j(y) \tag{60}$$

Multiplication between i-notation equation and $\Phi_j(y)$ and between j-notation equation and $\Phi_i(y)$, then, integrating them yields below equation:

$$\int_0^L \Phi_i^{iv}(y)\Phi_j(y)dy = \lambda_i\int_0^L \Phi_i(y)\Phi_j(y)dy \tag{61}$$

$$\int_0^L \Phi_j^{iv}(y)\Phi_i(y)dy = \lambda_j\int_0^L \Phi_j(y)\Phi_i(y)dy \tag{62}$$

If integrating left side of equation (61) and (62), gives

$$\left[\Phi_i'''(y)\Phi_j(y)\right]_0^L - \left[\Phi_i''(y)\Phi_j'(y)\right]_0^L + \int_0^L \Phi_i''(y)\Phi_j''(y)dy = \lambda_i\int_0^L \Phi_i(y)\Phi_j(y)dy \tag{63}$$

$$\left[\Phi_j'''(y)\Phi_i(y)\right]_0^L - \left[\Phi_j''(y)\Phi_i'(y)\right]_0^L + \int_0^L \Phi_j''(y)\Phi_i''(y)dy = \lambda_j\int_0^L \Phi_j(y)\Phi_i(y)dy \tag{64}$$

Subtracting between equation (63) and (64), obtained

$$(\lambda_i - \lambda_j)\int_0^L \Phi_i(y)\Phi_j(y)dy = 0 \tag{65}$$

This condition will be obtained if $i \neq j$ and $\lambda_i \neq \lambda_j$. So,

$$\int_0^L \Phi_i(y)\Phi_j(y)dy = 0 \tag{66}$$

Substituting equation (66) into equation (62) and (64), gives

$$\int_0^L \Phi_j^{iv}(y)\Phi_i(y)dy = 0 \text{ for } i \neq j \tag{67}$$

and

$$\int_0^L \Phi_i''(y)\Phi_j''(y)dy = 0 \text{ for } i \neq j \tag{68}$$

Equations (66), (67) and (68) show orthogonally of flexural vibration of beam. If condition $i = j$, so,

$$\int_0^L \Phi_i^2(y)dy = C_i \tag{69}$$

where C_i is any constant.

C. Natural Frequencies of Cantilever Beam

Writing equation (44) and (45) in matrix form, obtained

$$\begin{bmatrix} \cos(kL)+\cosh(kL) & \sin(kL)+\sinh(kL) \\ \sin(kL)-\sinh(kL) & -\cos(kL)-\cosh(kL) \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{70}$$

Hence, Equation (70) has a nontrivial solution if and only if the determinant of the coefficient matrix of C is zero; that is would gives

$$\det\begin{bmatrix} \cos(kL)+\cosh(kL) & \sin(kL)+\sinh(kL) \\ \sin(kL)-\sinh(kL) & -\cos(kL)-\cosh(kL) \end{bmatrix} = 0$$

$$-\cos^2(kL) - 2\cos(kL)\cosh(kL) - \cosh^2(kL) - \sin^2(kL) + \sinh^2(kL) = 0$$

$$\cos^2(kL) + \sin^2(kL) + 2\cos(kL)\cosh(kL) + \cosh^2(kL) - \sinh^2(kL) = 0$$

$$1 + 2\cos(kL)\cosh(kL) + 1 = 0$$

$$\cos(kL)\cosh(kL) + 1 = 0 \tag{71}$$

D. Mode Shapes of Cantilever Beam

The constants C in equation (53) are arbitrary. However, in order for the dynamic solution for the displacement to be equal to the static solution (at time t = 0 s), C value must fulfills for $\Phi(0)=0$ and $\Phi(L)=1$.

$$\Phi(L)=C\left\{\begin{matrix}[\cos(kL)-\cosh(kL)] \\ -\dfrac{\cos(kL)+\cosh(kL)}{\sin(kL)+\sinh(kL)}[\sin(kL)-\sinh(kL)]\end{matrix}\right\}$$

$$1=C\left[\frac{2\cos(kL)\sinh(kL)-2\cosh(kL)\sin(kL)}{\sin(kL)+\sinh(kL)}\right]$$

$$C=\frac{1}{2}\left[\frac{\sin(kL)+\sinh(kL)}{\cos(kL)\sinh(kL)-\cosh(kL)\sin(kL)}\right]$$

$$C=-\frac{1}{2} \tag{72}$$

Substituting equation (72) into equation (53), be have mode shapes equation

$$\Phi(y)=\frac{1}{2}\left\{\begin{matrix}\dfrac{\cos(kL)+\cosh(kL)}{\sin(kL)+\sinh(kL)}[\sin(ky)-\sinh(ky)] \\ -[\cos(ky)-\cosh(ky)]\end{matrix}\right\} \tag{73}$$

Rearranging equation (73), obtained

$$\Phi_n(y)=\frac{1}{2}\left\{\begin{matrix}[c_n\sin(k_n y)-\cos(k_n y)] \\ -[c_n\sinh(k_n y)-\cosh(k_n y)]\end{matrix}\right\} \tag{74}$$

where

$$c_n=\frac{\cos(k_nL)+\cosh(k_nL)}{\sin(k_nL)+\sinh(k_nL)} \tag{75}$$

3. Results and Discussion

The frequency equation can be solved for the constants, kL, the first five are shown below in Table 1 that obtained from graphic of equation (71) (See Figure 4).

Table 1. The first five natural frequency of cantilever beam.

No	kL	K	From equation (31)	
			$\omega^2=\dfrac{k^4EI}{\gamma_m}$	$\omega=k^2\sqrt{\dfrac{EI}{\rho A}}$
1	1.875	$\dfrac{1.875}{L}$	$\dfrac{12.3596EI}{L^4\gamma_m}$	$\omega=\dfrac{3.5156}{L^2}\sqrt{\dfrac{EI}{\rho A}}$
2	4.694	$\dfrac{4.694}{L}$	$\dfrac{485.4811EI}{L^4\gamma_m}$	$\omega=\dfrac{22.0336}{L^2}\sqrt{\dfrac{EI}{\rho A}}$
3	7.855	$\dfrac{7.855}{L}$	$\dfrac{3807.0165EI}{L^4\gamma_m}$	$\omega=\dfrac{61.7010}{L^2}\sqrt{\dfrac{EI}{\rho A}}$
4	10.996	$\dfrac{10.996}{L}$	$\dfrac{14619.7156EI}{L^4\gamma_m}$	$\omega=\dfrac{120.9120}{L^2}\sqrt{\dfrac{EI}{\rho A}}$
5	14.137	$\dfrac{14.137}{L}$	$\dfrac{39941.9287EI}{L^4\gamma_m}$	$\omega=\dfrac{199.8548}{L^2}\sqrt{\dfrac{EI}{\rho A}}$

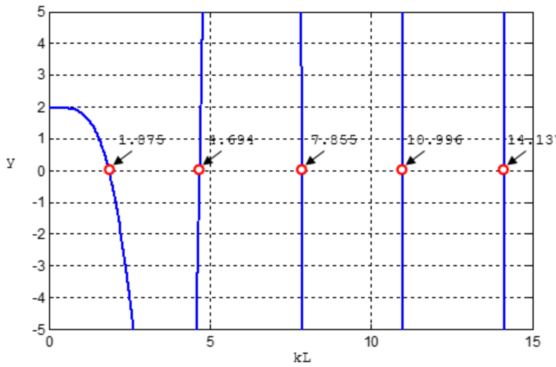

Figure 4. Graphic of equation (71).

First five mode shapes of a straight vertical cantilever beam at first five natural frequencies are showed by Figure 5.

For each frequency, there exists a characteristic vibration and rearranging equation (52), gives

$$u_n(y,t)=[A_n\cos(\omega t)+B_n\sin(\omega t)]\Phi_n(y) \tag{76}$$

or

$$u(y,t)=\sum_{n=1}^{\infty}\{[A_n\cos(\omega t)+B_n\sin(\omega t)]\Phi_n(y)\} \tag{77}$$

First derivatives of equation (77) is

$$\frac{\partial u}{\partial t}(y,t)=\sum_{n=1}^{\infty}\{[-\omega A_n\sin(\omega t)+\omega B_n\cos(\omega t)]\Phi_n(y)\} \tag{78}$$

Where A_n and B_n are constants which can be obtained from the initial conditions. For initial displacement:

$$u(y,0)=f(y) \tag{79}$$

and initial velocity:

$$\frac{\partial u}{\partial t}(y,0)=g(y) \tag{80}$$

Substituting initial displacement equation (79) into equation (77) yields

$$f(y)=\sum_{n=1}^{\infty}A_n\Phi_n(y) \tag{81}$$

A_n can be solved by

$$A_n=\frac{2}{L}\int_0^L f(y)\Phi_n(y)dy \tag{82}$$

Substituting initial velocity equation (80) into equation (78) yields

$$g(y)=\sum_{n=1}^{\infty}\omega B_n\Phi_n(y) \tag{83}$$

Thus, B_n can be solved by

$$B_n=\frac{2}{\omega L}\int_0^L g(y)\Phi_n(y)dy \tag{84}$$

A_n in equation (82) depends on the initial position at time t = 0 s, and B_n in equation (84) depends on the initial velocity. In this work, the beam starts its vibration when displaced and at rest. Thus, The value of B_n will be zero ($B_n = 0$).

The initial displacement, $u(y,0)$ was found above, equation (10). Hence,

$$A_n=\frac{2}{L}\int_0^L \frac{PL}{EI}\left(\frac{y^2}{2}-\frac{y^3}{6L}\right)\Phi_n(y)dy \tag{85}$$

where $\Phi_n(y)$ is mode shapes function (equation (74)). Thus,

$$A_n=\frac{2}{L}\int_0^L \frac{PL}{EI}\left(\frac{y^2}{2}-\frac{y^3}{6L}\right)\frac{1}{2}\left\{\begin{matrix}[c_n\sin(k_ny)-\cos(k_ny)]\\-[c_n\sinh(k_ny)-\cosh(k_ny)]\end{matrix}\right\}dy$$
$$=\frac{P}{EI}\int_0^L\left(\frac{y^2}{2}-\frac{y^3}{6L}\right)\left\{\begin{matrix}[c_n\sin(k_ny)-\cos(k_ny)]\\-[c_n\sinh(k_ny)-\cosh(k_ny)]\end{matrix}\right\}dy \tag{86}$$

Equations (87) below can be used to facilitate in integral equation (86) above.

$$a_1=\frac{c_n}{2}\int_0^L y^2\sin(k_ny)dy \tag{87.a}$$

$$a_2=-\frac{1}{2}\int_0^L y^2\cos(k_ny)dy \tag{87.b}$$

$$a_3=-\frac{c_n}{2}\int_0^L y^2\sinh(k_ny)dy \tag{87.c}$$

$$a_4=\frac{1}{2}\int_0^L y^2\cosh(k_ny)dy \tag{87.d}$$

$$a_5=-\frac{c_n}{6L}\int_0^L y^3\sin(k_ny)dy \tag{87.e}$$

$$a_6=\frac{1}{6L}\int_0^L y^3\cos(k_ny)dy \tag{87.f}$$

$$a_7=\frac{c_n}{6L}\int_0^L y^3\sinh(k_ny)dy \tag{87.g}$$

$$a_8=-\frac{1}{2}\int_0^L y^3\cosh(k_ny)dy \tag{87.h}$$

Furthermore, obtained

$$A_n=\frac{P}{EI}\sum_{i=1}^{8}a_i \tag{88}$$

Then, be have equation (89) and (90) as displacement and velocity equations of free vibration of cantilever beam respectively.

$$u(y,t)=\sum_{n=1}^{\infty}\left[\frac{P}{EI}\left(\sum_{i=1}^{8}a_i\right)\cos(\omega t)\Phi_n(y)\right] \tag{89}$$

$$\frac{\partial u}{\partial t}(y,t)=-\sum_{n=1}^{\infty}\left[\frac{P}{EI}\left(\sum_{i=1}^{8}a_i\right)\omega\sin(\omega t)\Phi_n(y)\right] \tag{90}$$

The natural frequency of cantilever beam was found above with $\omega=k^2\sqrt{\frac{EI}{\rho A}}$, so, equation (89) and (90) can be written respectively in form

$$u(y,t)=\sum_{n=1}^{\infty}\left[\frac{P}{EI}\left(\sum_{i=1}^{8}a_i\right)\cos\left(k^2\sqrt{\frac{EI}{\rho A}}\,t\right)\Phi_n(y)\right] \tag{91}$$

$$\frac{\partial u}{\partial t}(y,t)=-\sum_{n=1}^{\infty}\left[\frac{P}{EI}\left(\sum_{i=1}^{8}a_i\right)k^2\sqrt{\frac{EI}{\rho A}}\sin\left(k^2\sqrt{\frac{EI}{\rho A}}\,t\right)\Phi_n(y)\right] \tag{92}$$

Equation (91) and (92) show that flexural rigidity influences natural frequencies, displacement equation, and velocity equation of free vibration of cantilever beam.

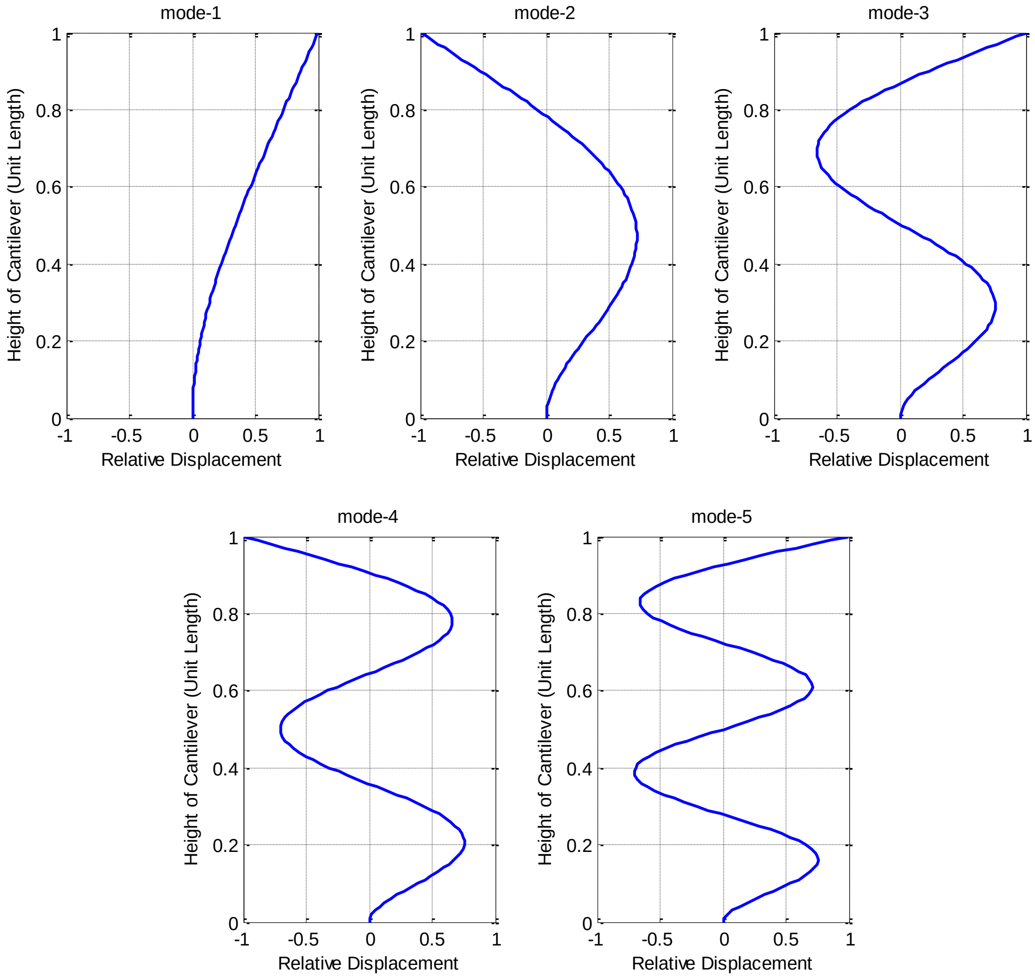

Figure 5. Mode shapes of a straight vertical cantilever beam.

The properties of cantilever beam that be analyzed in this work are showed in Table 2.

Table 2. Properties of cantilever beam.

No	Properties	Notation	Value	Unit
1	Elastic modulus	E	2.00×10^{11}	Pa
2	Moment of inertia	I	2.67×10^{-8}	m^4
3	Cross-section area	A	8.00×10^{-4}	m^2
4	Density	ρ	7862.30	kg/m^3
5	length	L	1.00	m
6	Poison ratio	υ	0.32	
7	Initial Load	P	100.00	N

Graphics of free vibration of cantilever beam with properties at Table 2 and P = 100 N can be seen at Figure 6.

4. Conclusion

From this work, can be concluded that mode shapes and natural frequencies of a straight vertical cantilever beams can be determined using solving partial differential equation by Fourier series. Separation method of variables is suitable to be used for the solution of free vibration of a straight vertical cantilever beams. The Fourier series, such Fourier cosine series and Fourier sine is suitable to be used for the solution of free vibration of a straight vertical cantilever beams. From case study of a straight vertical cantilever beams is obtained first five mode shapes, natural frequencies and simulation of free vibration. Simulation of a modeled cantilever beam with 2 GPa modulus of elasticity, 2.67×10^{-8} m^4 moment of inertia, 8×10^{-4} m^2 cross-section, 7862.30 kg/m^3 density, 1 m length, and 100 N initial load obtained first five natural frequencies respectively 16.29, 102.11, 285.95, 560.36, and 926.22 rad/s.

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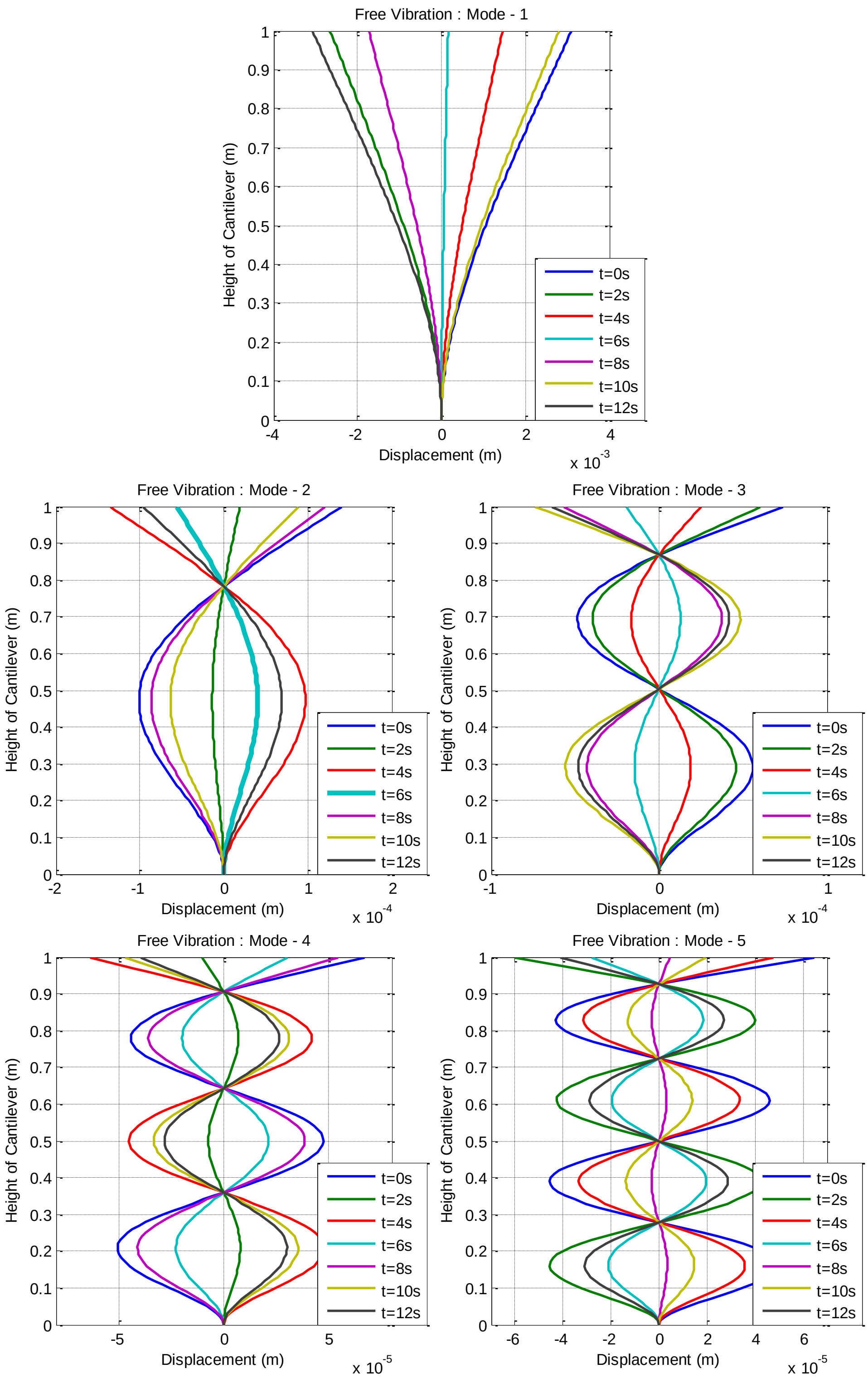

Figure 6. Graphics of free vibration of cantilever beam with properties at Table 2.

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