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A REGIONAL REFINEMENT FOR FINITE ELEMENT MESH DESIGN USING COLLAPSIBLE ELEMENT

Priyo Suprobo and Tavio
Lecturer, Department of Civil Engineering, Faculty of Civil Engineering & Planning,
Sepuluh Nopember Institute of Technology (ITS), Surabaya, Indonesia

ABSTRACT

A practical algorithm for automated mesh design in finite element analysis is developed. A regional mixed mesh improvement procedure is introduced. The error control, algorithm implementation, code development, and the solution accuracy are discussed. Numerical example includes automated mesh designs for plane elastic media with singularities. The efficiency of the procedure is demonstrated.

Keywords: regional refinement, mesh generation, isoparametric element, collapsible element

INTRODUCTION

Mesh generation and mesh improvement procedures have drawn considerable research interest in finite element development. This in part is due to their critical importance in the improvement of solution accuracy, particularly for stress analysis problems involving high stress and displacement gradients. For problems with stress singularities, such as concentrated applied loads, crack-rips and stress wave fronts, a finer mesh is usually required in a small region where stress concentration occurs [1].

Among the different strategies used for adaptive improvement of the element meshes, the following four are mentioned: (a) redeveloping meshes, (b) moving the nodes, (c) changing the order of the interpolation function, and (d) a mixture of the three basic methods. In the first strategy, the DOF of each element are kept constant and an improved mesh is constructed by optimizing the elements into small elements. This is usually referred to as the "h-method". In the second strategy, the quality of the finite element solutions is improved by optimizing the location of the nodes, while keeping the number of degrees of freedom fixed. This is usually referred to as the "r-method". The third strategy is based on increasing of the order of the polynomials of the element formulation and it is usually referred to as the "p-method" [2, 3].

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Quite often, a single method is not efficient for engineering practice. It is then natural to consider mixed approaches of refinement. Algorithms combining the "h- and p-methods" have been recently proposed, blending the two methods to provide a more flexible and computationally efficient procedure [4]. The performance was studied theoretically by Babuska and Dorr [5], Babuska and Rheinboldt [6], and Babuska and Rank [7]. The study also considered coupling mesh generations, cost estimates, error predictors and mesh refinements.

Las Casas [8] studied the combination of the r and h refinement method which was implemented for two dimensional elasticity problems. An initial improvement was obtained for coarse mesh using the r version to reach a near-uniform distribution of element error. A global refinement is then performed by equally subdividing all elements in the domain. Numerical examples showed that the use of r-h techniques provided a variable approach to generate efficient grids. The data structure is simple and requires reasonable computational effort. However, Las Casas considered 4-node elements only. When the p-refinement is performed for an element, all the adjacent elements must also be divided accordingly to maintain the compatibility. Thus, too many redundant sub-divisions are required. Furthermore, elements with a large aspect ratio may also appear and lead to possible numerical difficulties.

Further study in r-h refinement approach has been reported by Ning [9]. In this study, a collapsible isoparametric element was adopted.

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The benefit of using the collapsible elements is that the choice of four nodes to nine nodes in an element gives a smooth and compatible transition from a coarse mesh to a more refined one within a region. Numerical examples for stress concentration problems showed that the method provided convenience and efficiency.

In this study, attention is focused on the development of an adaptive mixed refinement algorithm by using collapsible isoparametric elements. Plane elements may be varied from four nodes to nine nodes. Furthermore, the global mesh can be subdivided into regions. Based on the accuracy criteria, each region can be refined differently. The use of collapsible elements ensures conformity between regions. An adaptive procedure can also be introduced to define the refinement zones.

Implementation of the concept of regional refinement is described in this study. Error estimates, indicators for improvement measures, are presented. Numerical examples are given to verify the concept and the algorithm.

REGIONAL REFINEMENT

In this section, the concept of regional refinement by using collapsible elements is presented. To illustrate the procedure, let the stress concentration region be small as compared to the entire plane elements. We may divide the mesh into several subregions as shown in Fig. 1. A uniform mesh may be adopted for the initial computation with r-refinements performed for the entire plane elements based on an accuracy criterion. Then, coordinates of the elements in the region far away the singular point are assumed to be fixed. Further h or p-refinements in the subregions near the crack may be introduced. R- refinements may then be performed for only part of the mesh. The procedure is repeated until the tolerance requirement is met.

The use of collapsible elements introduces considerable flexibility in handling the interface regions. Fig. 2 shows an example. The initial mesh is constructed by 4-node elements in a simple rectangular interior grid. After a set number of r-improvements, refined regions are defined; one may choose to simply impose a p-improvement using 9-node elements for the refined region and allowing the outside region remains unchanged.

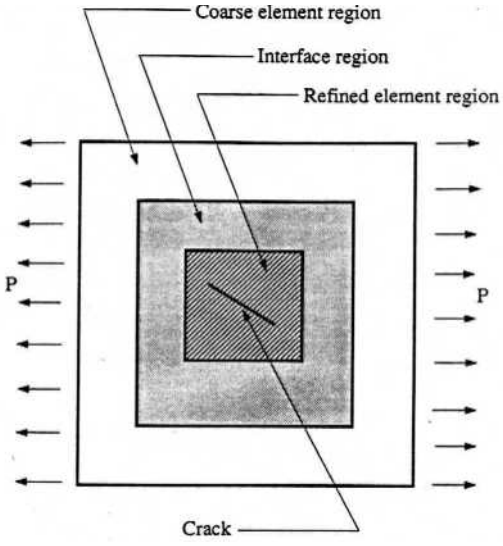


Figure 1. Regional Refinement for a Crack Problem

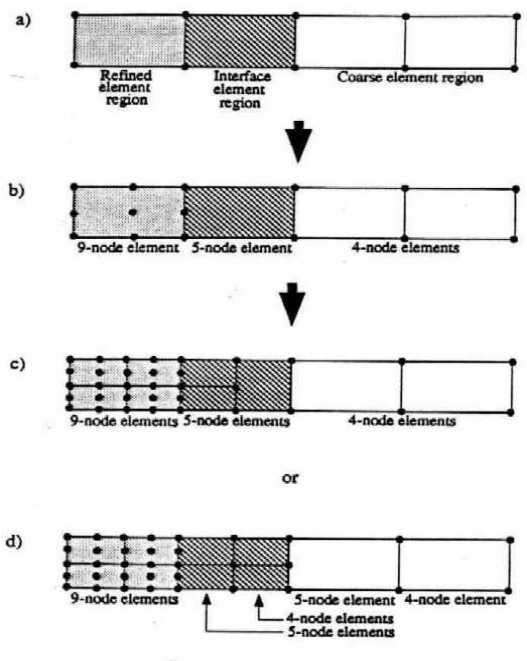


Figure 2. Regional Refinement with Collapsible Element

An automatic interface region is created by 5-node elements. Further improvements may be imposed to the refined region only. For example, a set of r-cycles may be imposed to relocate the coordinates of the nine-node elements, while the interface and coarse regions remain fixed. To introduce an hp-refinement in the refined region, one may simply subdivide the refined region uniformly, as shown in Fig. 2(c). The interface region is also subdivided into a multiple of 5-node elements. Alternatively, one may let the interface region advance into the other types of isoparametric elements. Fig. 2(d) shows the use of pairs of 4-node and 5-node elements. Another possibility is the use of a 6-

node and two 5-node elements in the interface region. Thus, the 4-node element to 9-node element may be used together in a complex mesh.

ERROR ESTIMATES

To implement the mixed refinement method, an important step is to evaluate the interpolation error in each element. Since exact solutions are not available, it is not possible to find the true error. Thus the refinement is based on an estimate of the interpolation error bound. A discussion of the error bound estimates can be found in Diaz [10]. The selection of Sobolev norm for 4-node isoparametric elements has been discussed by Las Casas [8] and for collapsible isoparametric elements by Ning [9].

Following Diaz [10], and Diaz, Kikuchi, Taylor [11], the discretization error bound has the form

$$\|e\| = \|u - u_h\|_m \le Ch^{k+1-m}|u|_{k+1} \quad (1)$$

where:

- u - exact solution
- u_h - finite element solution
- k - order of finite element interpolation function
- m - an integer, $0 \le m \le k$
- $|u|$ - the Sobolev semi-norm of u
- C - a constant
- h - diameter of the element

The Sobolev semi-norm for the solution domain Ω can be evaluated as

$$|u|_{k+1} = \left[\int_\Omega \sum_{|\alpha|\le k+1} \left|D^\alpha u\right|^2 d\Omega\right]^{\frac{1}{2}} \quad (2)$$

where:

$$D^\alpha u = \frac{\partial^\alpha u}{\partial^{\alpha_1} x\, \partial^{\alpha_2} y}$$

$$\alpha_1 + \alpha_2 = \alpha$$

Since the number of nodes and the edges with additional nodes are varied, the shape functions of a collapsible element are complicated. Table 1 shows the shape functions for a linear-quadratic collapsible element.

Error Bound in 4-Node Elements

As shown in Table 1, the highest order term is the "rs" term. Since the shape functions do not contain the other quadratic terms, the highest power of the polynomial in Eq. (1), *k*, is equal to 1. Taking $m = 0$, the error is bounded by

$$\|e\| \le Ch^2|u|_2 \quad (3)$$

In this inequality, second derivatives of the shape function are needed. However, the polynomials for the shape function have only C^1 continuity. Thus, it is not possible to evaluate the Sobolev norm which requires second derivatives.

Several approaches have been proposed in the literature to find the approximate Sobolev norm. The approach suggested by Diaz [10] and tested by Las Casas [8] and Ning [9] is adopted in the present study. The first derivatives of displacement are found at the assigned Gaussian points using a least square procedure to find the functional form of in $\partial u/\partial x$ the global coordinate system. The equation is

$$\frac{\partial u}{\partial x} = ax + by + c \quad (4)$$

The second derivative $\partial^2 u/\partial x^2 = a$ can then be calculated. Following the same technique, the other partial derivatives $\delta^2 u/\delta x\delta y$, $\delta^2 u/\delta y^2$, $\delta^2 v/\delta x^2$, $\delta^2 v/\delta x\delta y$, $\delta^2 v/\delta y^2$, can be calculated by finding the functional form of $\delta u/\delta y$, $\delta v/\delta x$, and $\delta v/\delta y$.

Error Bound for 5 to 9-Node Elements

The highest power of the shape function polynomial for a 5 to 9-node isoparametric element is two. Hence, the error bounded by the inequality for $k = 2$ and $m = 1$ is

$$\|e\| \le Ch^2|u|_3 \quad (5)$$

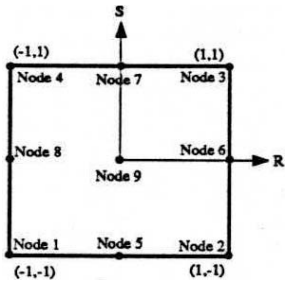
which requires the evaluation of third order partial derivatives. Since the second derivatives can be directly calculated from the shape functions, the same approach discussed previously may be used to find the third derivatives. Alternatively, we may fit the values of the first derivatives at Gaussian points to a quadratic function of x and y in the form

$$\frac{\partial u}{\partial x} = ax^2 + bxy + cy^2 + dx + ey + f \quad (6)$$

The $\partial^3 u/\partial x^3$ is equal to $2a$. We find that this approach seems to yield better results, since the first derivatives are a higher order of accuracy.

It is adopted in the present algorithm.

Table 1. Shape Function for 4-9 Nodes Collapsible Isoparametric Element



N_i	Function	N_1 to N_4	Add N_5	Add N_6	Add N_7	Add N_8	Add N_9
N_1	$\frac{(1-r)(1-s)}{4}$	N_1	$-\frac{N_5}{2}$			$-\frac{N_8}{2}$	$-\frac{N_9}{4}$
N_2	$\frac{(1+r)(1-s)}{4}$	N_2	$-\frac{N_5}{2}$	$-\frac{N_6}{2}$			$-\frac{N_9}{4}$
N_3	$\frac{(1+r)(1+s)}{4}$	N_3		$-\frac{N_6}{2}$	$-\frac{N_7}{2}$		$-\frac{N_9}{4}$
N_4	$\frac{(1-r)(1+s)}{4}$	N_4			$-\frac{N_7}{2}$	$-\frac{N_8}{2}$	$-\frac{N_9}{4}$
N_5	$\frac{(1-r^2)(1-s)}{2}$		N_5				$-\frac{N_9}{2}$
N_6	$\frac{(1+r)(1-s^2)}{2}$			N_6			$-\frac{N_9}{2}$
N_7	$\frac{(1-r^2)(1+s)}{2}$				N_7		$-\frac{N_9}{2}$
N_8	$\frac{(1-r)(1-s^2)}{2}$					N_8	$-\frac{N_9}{2}$
N_9	$(1-r^2)(1-s^2)$						N_9

INDICATORS FOR IMPROVEMENT MEASURES

Several indicators for improvements have been introduced to control the process of mesh improvement.

Potential Energy

The equilibrium state of the exact solution can be thought to be in a state of minimum total potential energy due to the external loading. Thus, an improvement of the element mesh should lead to a decrease of the total potential energy. For plane stress problems, the total potential energy can be calculated by:

$$\Pi = \sum_{\Omega} \frac{1}{2}\left(\varepsilon_{xx}\sigma_{xx} + \varepsilon_{yy}\sigma_{yy} + 2\varepsilon_{xy}\sigma_{yy}\right) - \sum\left(f_x u + f_y v\right) \quad (7)$$

In the algorithm, the value of Π is calculated after each nodal movement and each sub-division of the mesh. If the value increases after the improvement**,** it indicates that the procedure yields a worse solution and the process should be terminated.

Relative Error

A relative error measure is used to control r-refinements. Since the constant C in Eq. 1 is unknown, the true error can not be evaluated. However, we may compare the error bound of all elements. A parameter Errel to evaluate the relative error is defined as:

$$\text{Errel} = \frac{\text{Max}\left(\text{ErrorBound}\right)_{\Omega}}{\text{Min}\left(\text{ErrorBound}\right)_{\Omega}} \quad (8)$$

The criterion for the r-refinement is that the relative error should converge to 1. The criterion for nodal movement is to move the nodes to the higher error bound region. The new nodal positions are found by a relation, which is based on the direct mean value of relaxation:

$$x_i^{new} = x_i^{old} + \beta\left(\frac{\sum_{j=1}^{M} E_j x_j}{\sum_{j=1}^{M} E_j}\right) \quad (9)$$

where β is an adjustable factor. After some numerical experiments, β is taken as 1.0 for the interior points and 0.5 for the boundary nodes. Convergence seems to improve if the boundary nodes movements is slowing down. M is the total number of elements connecting to the specific nodes.

Element Aspect Ratio

The element aspect ratio has a significant effect on the accuracy of the finite element solution. This parameter can be set for the r and h improvements to avoid severe element distortion. A ratio of 1/3 appears to be a good ratio.

Element Area Ratio

Due to the use of Gauss Siedel iteration, which is adopted to solve the simultaneous equations, the algorithm converges slowly or may fail to converge when the stiffness matrix contains large and small terms at the same time. To avoid this problem, an element area ratio must

be set to terminate the h-refinement or to control the nodal movement in the r-improvement.

THE CODE DEVELOPMENT

The code development is based on a program MOVIDA-C, implemented earlier by Las Casas [8] and improved by Ning [9]. In the program a Gauss-Siedel iteration method is used to solve simultaneous equations. The advantages of this method are the stiffness matrix storage is small, and the matrix sorting work is reduced. To deal with problems involving strong singularities, such as crack problems, some modifications have to be made. Due to the strong singularity, the Gauss Siedel method requires a large number of iterations before reaching convergence [12].

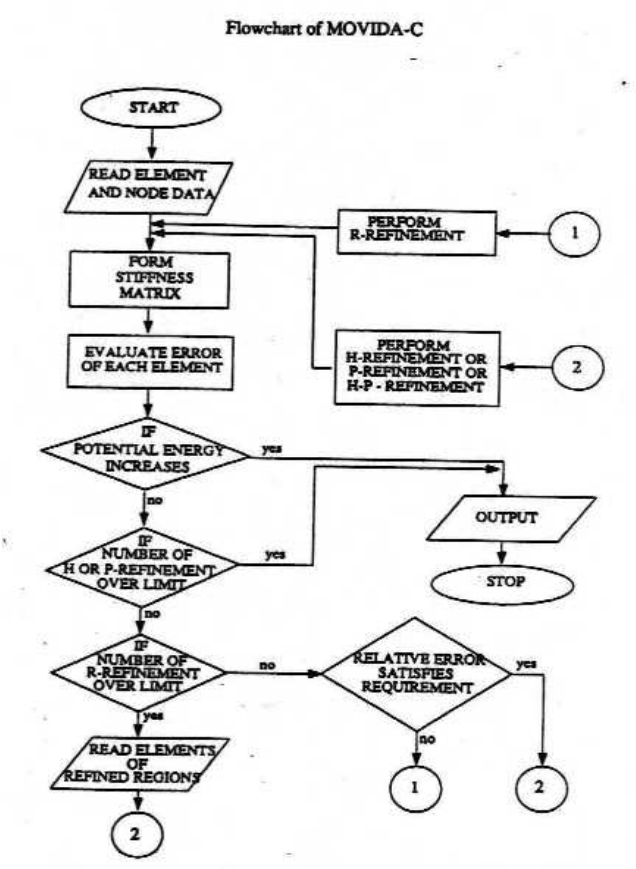


Figure 3. Flowchart of the Computer Code

This inefficiency is particularly noticeable after each h-refinement. To improve the rate of convergence, displacement values obtained for the previous mesh are used for the subsequent iteration. For the newly added nodes, interpolation values based on previous solutions are adopted This modification reduces the required number of iterations considerably. A flowchart of the computer code is given in Fig. 3.

NUMERICAL EXAMPLE: PLATE WITH AN ELLIPTIC HOLE

A square plate with an elliptic hole in the center is considered. The plate is 1 in (25.4 mm) thick and is made of a material with a Young's modulus of 3×10^7 psi (2.068×10^5 MPa) and a Poisson's ratio of 0.3. The distributed load, P, is 1000 lbs./in (1.751×10^{-1} N/mm). The analytical solution for an elliptic hole assumes an infinite plate which ignores the plate size effect. For the specified configuration shown in Fig. 4, σ_{xx} along the Y-axis has the form:

$$\sigma_{xx} = \sigma_0 \frac{3\cosh(\xi) - 2\sinh(\xi)}{\sinh(\xi)} \qquad (10)$$

where σ_0 is stress at infinity and y = 14.14 $\cosh(2\xi)$. ξ is an elliptic coordinate with $0.34 \le \xi \le 2.644$ for the present case. The maximum stress concentration occurs at top of the elliptic hole where σ_{xx} is equal to 7 σ_0. To illustrate the mesh improvement process, an initial mesh with 16 elements and 25 nodes is proposed, shown in Fig. 5(a). The first consideration is to improve the mesh by imposing 50r-refinements. The values of the potential energy and the relative error for the first ten, 20th and 50th steps are listed in the Table 2.

It is seen that both the relative error and the potential energy decrease slowly and both values almost become constant. Thus, it is logical to terminate further r-refinement after the 3rd iteration. Fig. 5(b) shows the finite element mesh after 3r-refinements. The next step of improvement is to impose a regional p-refinement near the hole. Elements 3, 4, 7, 8, 11, 12, 15 and 16 remain the same, and elements 1, 2, 5, 6, 9, 10, 13 and 14 are improved to 9-node elements (Fig. 5c). The total number of nodes becomes 55 nodes. The relative error after p-refinement becomes high; however, the potential energy decreases rapidly. To reduce the relative error, r-refinement is introduced again. Table 3 shown the improvement criterion values for the r-iterations. The potential energy reaches a constant value after the first iteration. Moreover, the minimum value of the relative error occurs in the first iteration also. After that, the relative error increases. Therefore, the r-refinement is only required once. Fig. 5(c) (16 elements and 55, nodes) shows the mesh improvement after

3r-p-lr.

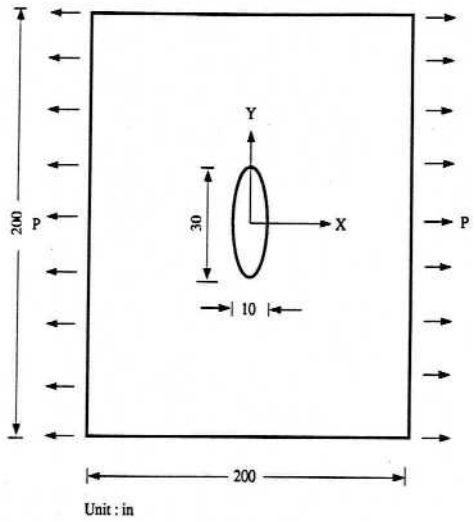


Figure 4. Square Plane, 200 × 200 in^2 (5080 × 5080 mm^2) with an Elliptic Hole, 30 in (762 mm) - 10 in (254 mm)

Table 2. Relative Error and Potential Energy for Plate with an Elliptic Hole (50 refinement)

Step	Relative Error	Potential Energy	
		lb-in	Nmm
Init, Mesh	82.38719	-0.17200E+03	-0.194324E+05
1r	28.30731	-0.17210E+03	-0.194437E+05
2r	28.37613	-0.17223E+03	-0.194584E+05
3r	27.13404	-0.17221E+03	-0.194561E+05
4r	30.10692	-0.17227E+03	-0.194629E+05
5r	27.85783	-0.17222E+03	-0.194573E+05
6r	31.75316	-0.17230E=03	-0.194663E+05
7r	29.67274	-0.17226E+03	-0.194618E+05
8r	32.73042	-0.17234E+03	-0.194708E+05
9r	31.22723	-0.17230E+03	-0.194663E+05
10r	32.59776	-0.17237E+03	-0.194742E+05
20r	10.72613	-0.17255E+03	-0.194946E+05
50r	6.56543	-0.17247E+03	-0.194855E+05

For the next mesh improvement, we only consider the elements in a smaller refinement region. In this step, a mixed of h and p refinements is applied. Elements 1, 5, 9 and 13 are then changed into four 9-node elements. Elements 2, 6, 10 and 14 become transition elements and are a combination of 5-node and 6 node isoparametric elements. r-improvement cycles are again implemented for the small refinement region and values of the indicators are evaluated. The total potential energy does not increase much after the first of r-refinement. Hence if an increasing total potential energy is adopted as a criterion to terminate further improvements, 3r-p-1r-hp-1r will yield the final mesh for the problem. The final mesh gives $\sigma_{xx \text{ at } (0,15)} = 6.577\ \sigma_0$ The closed form solution for an infinite plate is 7.0 σ_0. The improved mesh after 3r-p-1r-hp-1r step and the stress distribution results are shown in Fig. 5(d) and Fig. 6, respectively. The final mesh has 36 elements and 113 nodes.

Table 3. Relative Error and Potential Energy for Plate with an Elliptic Hole (3r-p-1r-hp-5r refinement)

Step	No's of Elem./ Nodes	Relative Error	Potential Energy	
			lb-in	Nmm
Init. Mesh	16/25	82.38719	-0.17200E+03	-0.194324E+05
3r	16/25	27.13404	-0.17217E+03	-0.194516E+05
3r-p	16/55	35.87261	-0.17207E+03	-0.194403E+05
3r-p-1r	16/55	4.88571	-0.17337E+03	-0.195872E+05
3r-p-2r	16/55	6.55262	-0.17337E+03	-0.195872E+05
3r-p-3r	16/55	5.42558	-0.17337E+03	-0.195872E+05
3r-p-4r	16/55	8.07788	-0.17337E+03	-0.195872E+05
3r-p-5r	16/55	6.37224	-0.17337E+03	-0.195872E+05
3r-p-1r-hp	36/133	11.84905	-0.17370E+03	-0.196245E+05
3r-p-1r-hp-1r	36/133	8.89900	-0.173370E+03	-0.195872E+05
3r-p-1r-hp-2r	36/133	10.93108	-0.173370E+03	-0.195872E+05
3r-p-1r-hp-3r	36/133	9.58200	-0.173370E+03	-0.195872E+05
3r-p-1r-hp-4r	36/133	13.37915	-0.173370E+03	-0.195872E+05
3r-p-1r-hp-5r	36/133	13.120073	-0.173370E+03	-0.195872E+05

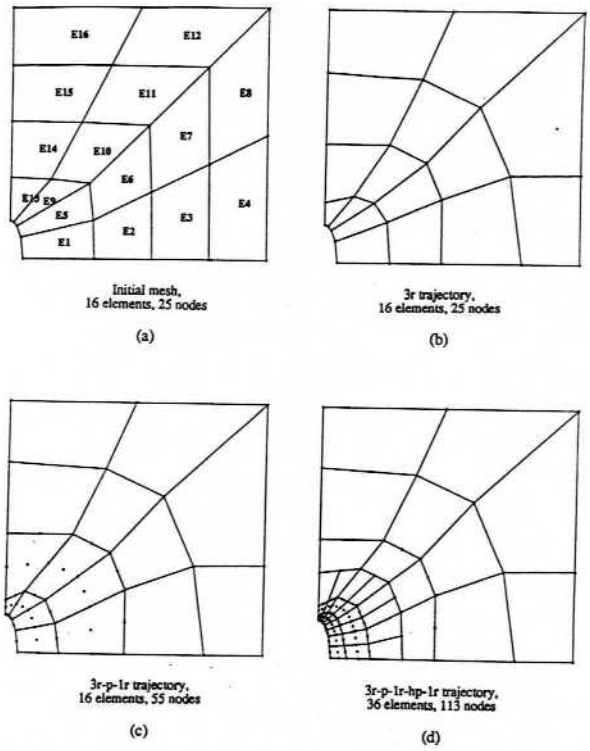


Figure 5. Improved Meshes

Another possibility is to use a combination of 4-node and 5-node isoparametric elements in the interface region. This may give another advantage in the finite element code development because there are only three types of isoparametric elements used: 4, 5, and 9 node iso-

parametric elements. After a similar 3r-p-1r-hp-1r improvement step, the finite element mesh has 40 elements and 113 nodes, shown in Fig. 7. With the same number of degrees of freedom in the final mesh, the stress concentration factor obtained is 6.596. The stress distributions during the improvement steps are shown in Fig. 9. The results are nearly identical to those previously obtained.

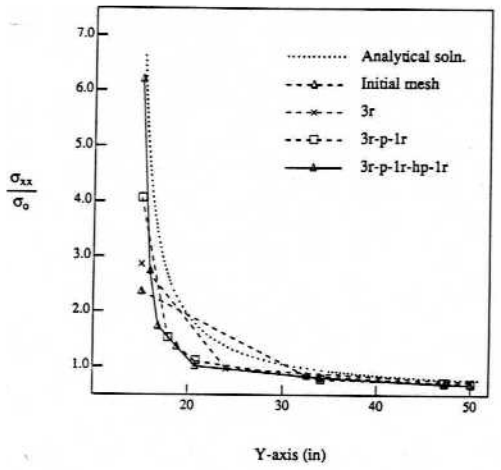


Figure 6. σ_{xx}/σ_0 Along the Y-axis (36 Elements, 113 Nodes), 1 in = 25.4 mm

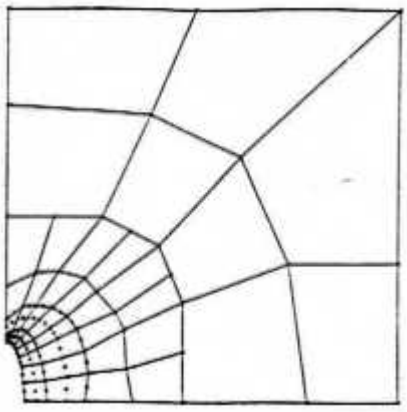


Figure 7. Improved Mesh (40 Elements, 113 Nodes, 3r-p-1r-hp-1r Trajectory)

To examine the efficiency and effectiveness of the regional refinement, other mixed improvement methods are applied to solve the same problem First the problem is solved by a semi-global rph-mixed refinement. A global p-refinement is used for improving the mesh with all elements changed to 9-node elements after 3r-refinement. Then, a regional refinement is introduced to the elements near the stress concentration. Elements 1, 5, 9 and 13 are improved to four 9-node elements and elements 2, 6, 10 and 14 are used as transition elements. Each of these transition elements contain two 9-node elements and one 6-node element. The final improved mesh is shown in Fig. 8, which has 36 elements and 139 nodes. The stress distribution along the Y-axis is shown in Fig. 10, which gives a maximum stress ratio 6.613. The stress concentration factor of the global and regional refinements is improved only 0.55% yet require 20% more computational time.

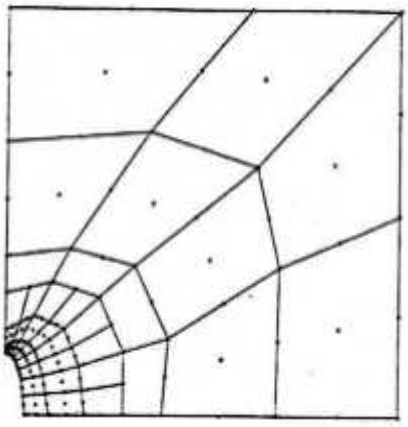


Figure 8. Improved Mesh in Semi-Global Refinement (36 Elements, 139 Nodes, 3r-p-1r-hp-1r Trajectory)

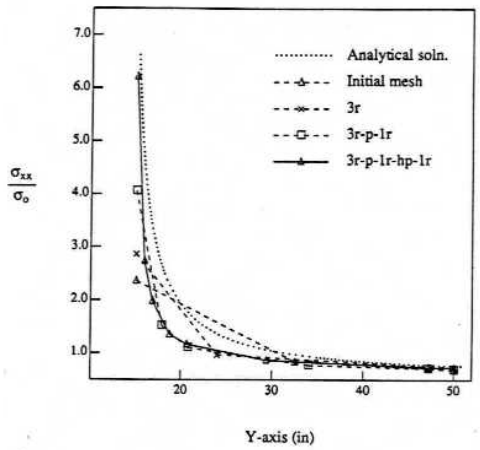


Figure 9. σ_{xx}/σ_0 Along the Y-axis (40 Elements, 113 Nodes), 1 in = 25.4 mm

Completely global h and p-refinements are also considered. Meshes for the global p-hp and p-hp-hp trajectories are shown in Fig. 11 and Fig. 12, respectively. Both improved meshes yield rather poor results. The compared results and execution times for all the improvement cases are listed in Table 4.

It is shown that the regional refinements give better solutions than the global refinements. Figures 13 and 14 show the stress distribution

and the improvement stress concentration value. In the process of the mesh improvements, the regional refinement consistently yields better solutions. Furthermore, in terms of the total number of degrees of freedom required, the advantage of the regional refinement is more evident as seen in Fig. 14. Compared to the analytical solution, the mesh based on the global p-hp-hp procedure yields a 29% difference; otherwise, the regional refinement gives 6% difference.

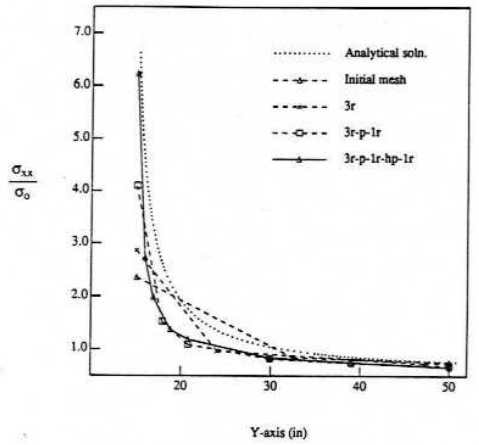


Figure 10. σ_{xx}/σ_0 Along the Y-axis (36 Elements, 136 Nodes), 1 in = 25.4 mm

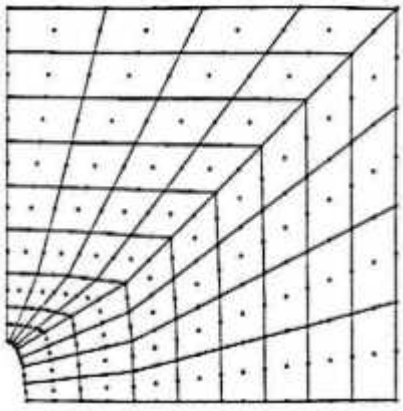


Figure 11. Improved Mesh in Global Refinement (64 Elements, 289 Nodes, p-hp Trajectory)

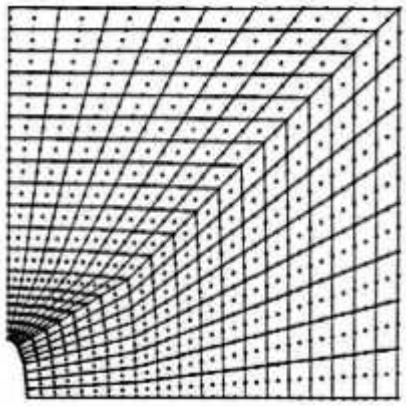


Figure 12. Improved Mesh in Global Refinement (258 Elements, 1089 Nodes, p-hp-hp Trajectory)

Table 4. Results and execution time for plate with an elliptic hole

Step	No's of Elem./ Nodes	Potential Energy		σ_{xx}/σ_0 at (0,15)	CPU time (second)
		lb-in	Nmm		
3r-p-1r-hp-1r	36/113	-0.17373E+03	-0.196279E+05	6.577	22.00
3r-p-1r-hp-1r	40/113	-0.17389E+03	-0.196460E+05	6.596	23.38
3r-p-1r-hp-1r	36/139	-0.17380E+03	-0.196358E+05	6.613	26.57
p	16/81	-0.17317E+03	-0.195646E+05	2.982	6.75
p-hp	64/289	-0.17347E+03	-0.195646E+05	4.068	39.68
p-hp-hp	256/1089	-0.17365E+03	-0.196188E+05	4.959	476.83

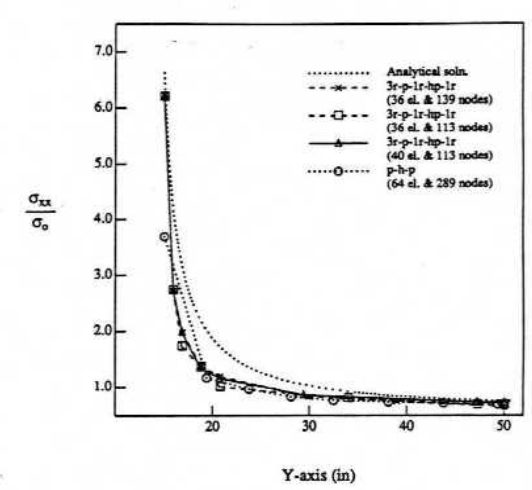


Figure 13. Comparison of σ_{xx}/σ_0 Along the Y-axis for Regional and Global Improvement Cycles, 1 in = 25.4 mm

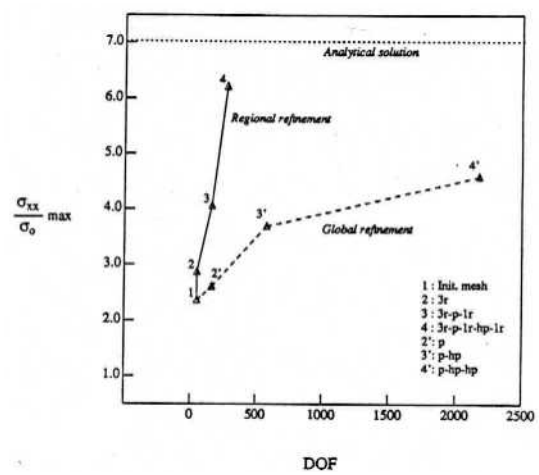


Figure 14. Comparison of Maximum σ_{xx}/σ_0 in Terms of DOF Used in the Regional and Global Refinements

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