DEVELOPMENT OF FINITE DIFFERENCE METHOD APPLIED TO CONSOLIDATION ANALYSIS OF EMBANKMENTS

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ABSTRACT

This study presents the development of the finite difference method applied to consolidation analysis of embankments. To analyse the consolidation of the embankment as real as possible, the finite difference method in two dimensional directions was performed. Existing soils under embankments have varying stresses due to stress history and geological background. Therefore, Skempton's parameter "*A*" which is a function of vertical stresses was taken into account in this study. Two case studies were chosen to verify the proposed method. It is found out that results obtained from the proposed method agree with either recorded data or results solved using another solution.

Keywords: Embankment, Excess pore pressure, Consolidation, Finite Difference method, Skempton's parameter

INTRODUCTION

Large settlements are often found to be one of the most serious problems of the highway embankments built over soft compressible soils. The largest component of the total settlement is consolidation. The consolidation is defined as "*a decrease in water content of a saturated soil without replacement of water by air*" [1]. Thus, it is essential to estimate the magnitude and time rate of consolidation before designing and constructing a highway embankment.

The conventional one dimensional consolidation theory developed by Terzaghi in 1925 [2] is commonly used to predict the magnitude of settlement and time rate of consolidation. Problems having nearly similar assumptions adopted in the Terzaghi theory can be predicted accurately. Such assumptions are homogeneous and fully saturated soil, incompressible solid particles and water, one dimensional flow and compression, small strains in soil, validity of Darcy's law, constant coefficient of permeability and coefficient of volume compressibility and unique relationship between void ratio and effective stress which is independent of time. However, problems in which the above assumptions are not relevant may not be predicted very well using the Terzaghi theory.

For an example, in the case of a highway embankment, dissipation of excess water pressures occurs in two directions, i.e., vertical and horizontal directions. The pore water pressures predicted with the two dimensional theory of consolidation is better than those predicted with the one dimensional one.

Furthermore, existing soils under the embankment have varying stresses due to stress history and geological background. This gives varying overconsolidation ratios in the existing soils under the embankment. In other words, Skempton's parameters "*A*" of the soils vary as they are functions of the overconsolidation ratios and the applied stresses at any points. The Terzaghi consolidation theory does not include this assumption.

This study presents the use of finite difference method applied to embankment problems. To support the limitation of the Terzaghi theory, two dimensional consolidation was adopted here. Skempton's parameter "*A*" which depends on the overconsolidation ratios and the applied vertical stresses was also considered.

THEORETICAL BACKGROUND

Rate of Consolidation

It is often found that the observed rate of settlement of embankment is faster than the predicted one obtained from one dimensional consolidation theory. The reason is that the

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width of the embankments is much smaller than the depth of the supporting soils under the embankments. In addition, the permeability of the supporting soils in the horizontal directions is usually larger than that in the vertical directions [3].

To improve the limitation of the one dimensional consolidation theory, it is necessary to adopt the two dimensional consolidation theory. Rendulic in 1937 [4] introduced the governing equation of the two dimensional consolidation theory as follows:

$$\frac{du}{dt} = C_h \frac{\partial^2 u}{\partial x^2} + C_v \frac{\partial^2 u}{\partial z^2} \tag{1}$$

where
u = excess pore water pressure
C_h = coefficient of consolidation in horizontal direction
C_v = coefficient of consolidation in vertical direction
x = horizontal coordinate
z = vertical coordinate

Equation (1) can be used to calculate one dimensional strain in conjunction with dissipation of pore pressures in two directions. Values of C_h and C_v can be obtained from the standard oedometer or Rowe cell tests.

In order to calculate the degree of consolidation precisely, it is important to divide the soil medium into a number of sublayers. These sublayers are usually based on the availability of different types of soils. Another objective to divide into this sublayers is to consider the stress distribution induced by the applied load. The stress decreases with depth according to the Boussinesq principle.

The correlation between the degree of consolidation and the pore water pressure at a given time t can be defined as:

$$U_t = 1 - \frac{\sum_{i=1}^{H} u_t}{\sum_{i=1}^{H} u_0} \tag{2}$$

where:
U_t = average degree of excess pore pressure dissipation at time t
u_t = excess pore water pressure at time t
u_o = initial excess pore water pressure
H = depth of soil medium

Finite Difference Method

The finite difference method has been applied to several geotechnical engineering problems [5]. This method has been proven accurate enough as long as the model meets the standard requirements. There are several ways to simplify a differential equation such as the forward, backward and central difference approximations. The basic idea of the difference approximations is a Taylor series expansion. The first derivative of a function *f(w)* with respect to *w* can be written as:

$$\frac{d(f(w))}{dw} = \frac{f(w+\Delta w) - f(w)}{\Delta w} \tag{3}$$

while the second derivative of a function *f(w)* with respect to *w* can be written as:

$$\frac{d^2(f(w))}{dw^2} = \frac{f(w+\Delta w) - 2f(w) + f(w-\Delta w)}{(\Delta w)^2} \tag{4}$$

Referring to the basic finite differential equation in equation (1), both equations (3) and (4) are applied. It involves the first derivative of u with respect to t in the left hand side and the second derivative of u with respect to both x and z in the right hand side.

The common method usually used to solve problems in two dimensional space is the explicit finite difference method. Although this method is easy to be implemented, it is not unconditionally stable. Therefore, the time interval and width of the finite difference grids should be selected so that the solution becomes unique and stable.

Alternatively, an implicit finite difference method which is well known as the Alternating Direction Implicit (ADI) method can be used to solve problems in two dimensional space. Remson et al. [6] adopted the ADI method which has advantages in solving finite different equations. The advantages are:
1. It is convergent and unconditionally stable.
2. It has second order accuracy in both space and time.
3. It leads to a tridiagonal system of equations.

Due to the above advantages, the ADI method has been chosen to analyse the consolidation problems of embankments in this study.

Figure 1 shows the point numbering convention used in the finite difference grid at time t. The lateral boundaries in an embankment problem are actually infinity. However, in modelling the embankment problem using finite difference grids, a certain distance from the vertical axis of symmetry (=L) can be taken as a lateral boundary as long as such a distance is quite far. Figure 2 shows finite difference grids of an embankment.

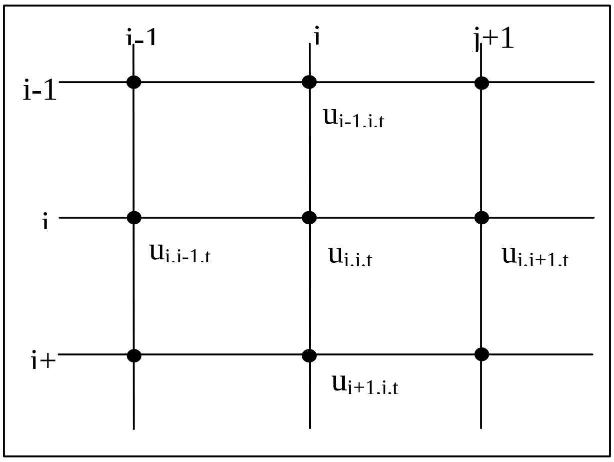

Figure 1. Point Numbering Convention on Finite Difference Grid

The ADI method required to solve equation (1) at node *i,j* from the n^{th} time level to the $(n+2)^{th}$ time level can be explained as follows [7]:

- for the time interval between *n* and (*n*+1):

$$\frac{u_{i,j,n+1}-u_{i,j,n}}{\Delta t}=C_h\frac{u_{i,j-1,n+1}-2u_{i,j,n+1}+u_{i,j+1,n+1}}{(\Delta x)^2}+ C_v\frac{u_{i-1,j,n}-2u_{i,j,n}+u_{i+1,j,n}}{(\Delta z)^2} \tag{5}$$

- for the time interval between (*n*+1) and (*n*+2):

$$\frac{u_{i,j,n+2}-u_{i,j,n+1}}{\Delta t}=C_h\frac{u_{i,j-1,n+1}-2u_{i,j,n+1}+u_{i,j+1,n+1}}{(\Delta x)^2} + C_v\frac{u_{i-1,j,n+2}-2u_{i,j,n+2}+u_{i+1,j,n+2}}{(\Delta z)^2} \tag{6}$$

Let

$$r_x=\frac{C_h\Delta t}{(\Delta x)^2} \tag{7}$$

$$r_z=\frac{C_v\Delta t}{(\Delta z)^2} \tag{8}$$

Equations (7) and (8) are substituted into equations (5) and (6). The unknown values for the excess pore water pressure are then arranged on the left hand side while the known values are put on the right hand side. Thus,

- for the time interval between *n* and (*n*+1):

$$-r_x u_{i,j-1,n+1}+(1+2r_x)u_{i,j,n+1}-r_x u_{i,j+1,n+1} = r_z u_{i-1,j,n}+(1-2r_z)u_{i,j,n}+r_z u_{i+1,j,n} \tag{9}$$

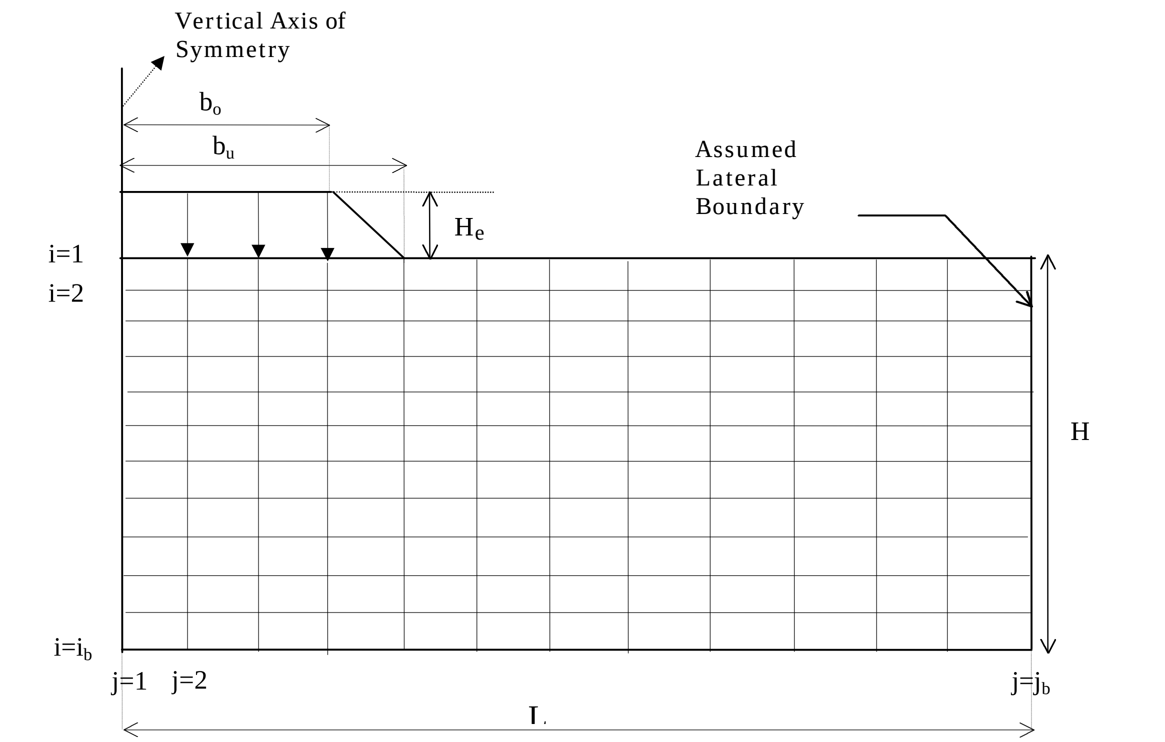

Figure 2. Finite Difference Grids of Embankment

- for the time interval between (n+1) and (n+2):

$$-r_z u_{i-1,j,n+2} + (1+2r_z)u_{i,j,n+2} - r_z u_{i+1,j,n+2} = r_x u_{i,j-1,n+1} + (1-2r_x)u_{i,j,n+1} + r_x u_{i,j+1,n+1} \quad (10)$$

It can be seen that three unknown values of excess pore water pressure are found in either equation (9) or equation (10). In equation (9), all the unknown values of excess pore water pressure lie on a horizontal line for a certain odd time step of each time cycle. This means that this equation is implicit in the horizontal direction. Hence, the pore pressure at all points along a horizontal line at the time level (n+1) can be calculated by solving the system of simultaneous equations referring only to the horizontal line of nodes. Similarly, equation (10) is implicit in the vertical direction for a certain even time step of each time cycle. The excess pore water pressure at all points along a vertical line at the time level (n+2) can be calculated by solving the system of simultaneous equations referring only to the vertical line of nodes.

The following matrix equation presents the system of simultaneous equations described in the previous paragraph [7].

$$[f]\{u\} = \{p\} \quad (11)$$

Matrices $[f]$, $\{u\}$ and $\{p\}$ are described as follows:

$$[f] = \begin{bmatrix} 1+2r_x & -2r_x & 0 & . & 0 & 0 & 0 \\ -r_x & 1+2r_x & -r_x & . & 0 & 0 & 0 \\ 0 & -r_x & 1+2r_x & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & -r_x & 1+2r_x & -r_x \\ 0 & 0 & 0 & . & 0 & -r_x & r_x' \end{bmatrix} \quad (12)$$

where:

$$r_x' = 1 + 2r_x - r_x . e^{1-(x_{jb}+1/x_{jb})} \quad (13)$$

$$\{u\} = \begin{Bmatrix} u_{i,1,n+1} \\ u_{i,2,n+1} \\ u_{i,3,n+1} \\ . \\ . \\ u_{i,j_{b-1},n+1} \\ u_{i,j_b,n+1} \end{Bmatrix} \quad (14)$$

$$\{p\} = \begin{Bmatrix} r_z u_{i-1,1,n} + (1-2r_z)u_{i,1,n} + r_z u_{i+1,1,n} \\ . \\ . \\ . \\ . \\ . \\ r_z u_{i-1,j_b,n} + (1-2r_z)u_{i,j_b,n} + r_z u_{i+1,j_b,n} \end{Bmatrix} \quad (15)$$

The unknown values of excess pore water pressure or $\{u\}$ are solved using the tridiagonal algorithm elimination method proposed by Remson et. al. [6].

Skempton's Pore Pressure Parameter, *A*

Skempton in 1954 [8] stated that the excess pore water pressure due to external loading can be expressed as:

$$\Delta u = B(\Delta s_3 + A(\Delta s_1 - \Delta s_3)) \quad (16)$$

where:

Δu = change in excess pore water pressure
B = Skempton's pore pressure parameter
Δs_3 = change in minimum principal stresses
A = Skempton's pore pressure parameter
Δs_1 = change in maximum principal stresses

Skempton's pore pressure parameter B depends on the degree of saturation of soil. In fully saturated soil, B is equal to unity.

Skempton's pore pressure parameter A depends on the magnitude of the preconsolidation load to which the soil has been subjected in the past. Values of A for different soils are shown in table 1.

Table 1. Values of Pore Pressure Coefficient *A* for Different Soils [8]

Type of Soil	*A* at failure
Clay with high sensitivity	0.75 to 1.50
Normally consolidated clay	0.50 to 1.00
Lightly overconsolidated clay	0.00 to 0.50
Heavily overconsolidated clay	-0.50 to 0.0
Compacted sandy clay	0.25 to 0.75
Compacted clay gravels	-0.25 to 0.25

Figure 3 shows the relationship between A and Over-Consolidation Ratio (OCR) at failure obtained from triaxial experiments.

The relationship between A and OCR can be expressed as [9]:

$$A = 0.927307 - 2.36593\,log(OCR) + 1.56293\,log^2(OCR) - 0.38556\,log^3(OCR) \quad (17)$$

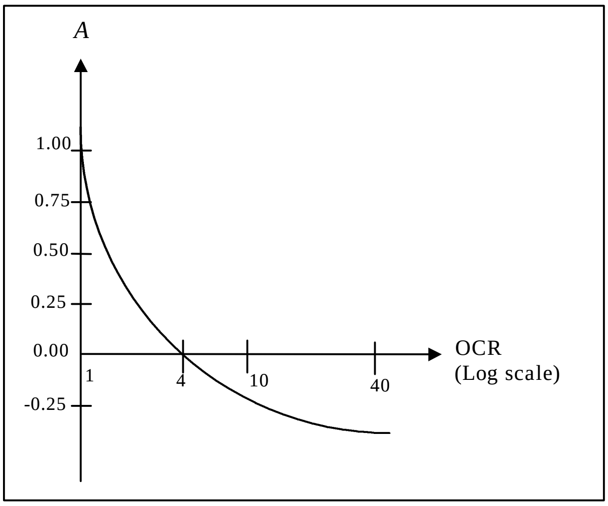

Figure 3. Pore Pressure Coefficient *A* versus OCR (failure condition)

Stress Distribution Due to Embankment Loading

To calculate stress distribution at a certain point in the soil medium due to embankment loading is based on the method of superposition of the stresses at that point due to symmetrical vertical triangular loading as shown in figure 4. The stresses at *A* due to embankment loading (*q*) are equal to the stresses at A due to the triangular loading in figure 4(b) minus the stresses at A due to the triangular loading shown in figure 4(c). The following equation developed by Gray (1936) and reported by Davis and Poulos [10] was adopted in this study.

$$s_z = \frac{q}{p}\left((a_1 + a_2) + \frac{x}{c}(a_1 - a_2)\right) \quad (18)$$

$$s_x = \frac{q}{p}\left((a_1 + a_2) + \frac{x}{c}(a_1 - a_2) - \frac{2z}{c}\ln\frac{R_1 R_2}{R_0^2}\right) \quad (19)$$

For the case where there are a number of soil layers, the method of superposition can also be adopted.

NUMERICAL RESULTS

Verification of the Proposed Method

The proposed method was verified using an embankment problem which has been solved using charts developed by Dunn and Razouki [7]. The embankment problem is as shown in figure 2. The data used for this problem were as follows:

* Drainage condition is permeable at the soil surface and impermeable at the base of the soil layer
* b_0 = 4 meter
* b_u = 5 meter
* H = 5 meter
* C_v = 0.84 m²/year
* C_h = 84 m²/year
* E = 21MN/m²
* γ soil = 16 kN/m³
* γ fill = 15 kN/m³
* H_e = 2 m

As shown in figure 5 the degree of consolidation obtained from the proposed method agreed with that obtained from the charts developed by Dunn and Razouki [9].

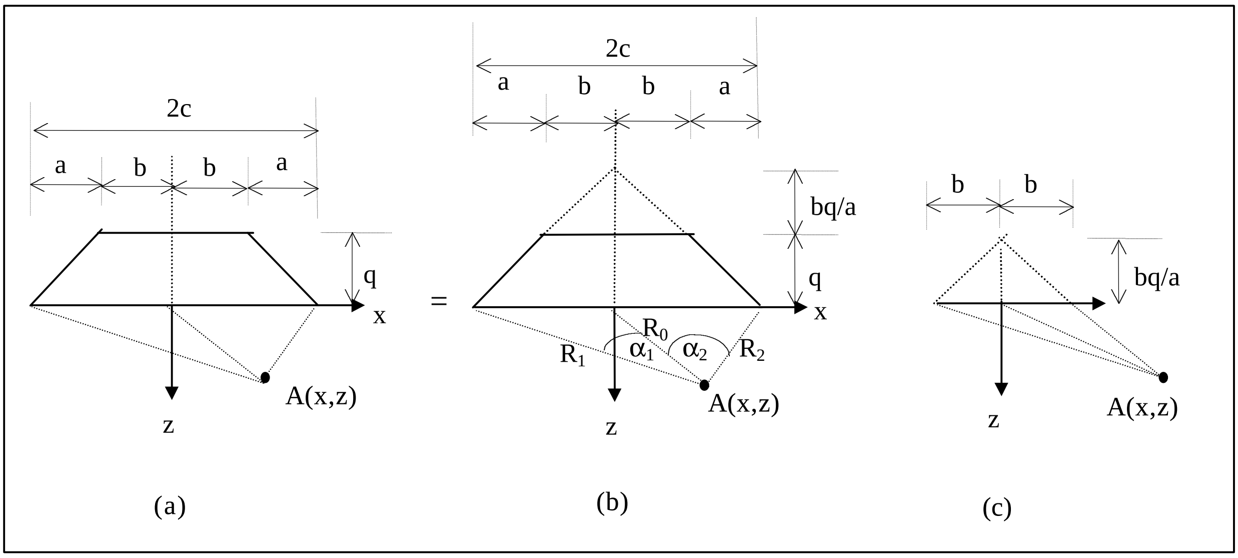

Figure 4. Stresses Due to Embankment Loading

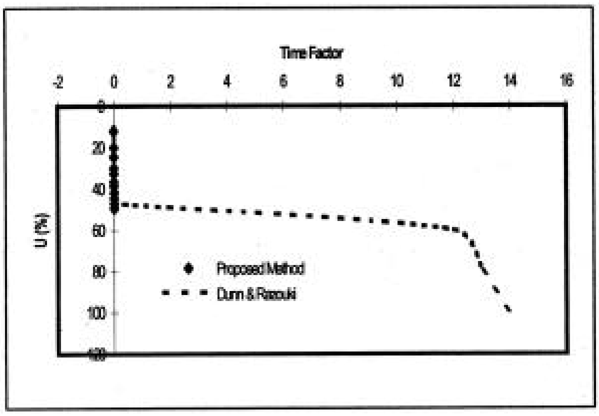

Figure 5. Comparison of Degree of Consolidation versus Time Factor

Excess Pore Water Pressure Dissipation of Trial Embankment at Panci Project

The trial embankment located at Km 37+045 of The Panci Toll highway near Bandung city - Indonesia was chosen as a study case to verify the proposed method. The layout of the trial embankment section at Sta 37+045 is shown in figure 6. The embankment was built in a number of stages. The final height of the test embankment was about 4.8 m.

There was six soil layers beneath the trial embankment. The soil layers from the top were the weathered zone, soft clay, desiccated layer, soft clay, desiccated layer and soft clay. The bottom layer was permeable. The soil properties of each layer are tabulated in Table 2 [9].

The C_v and C_h values used to simulate the excess pore water pressure dissipation behaviour were initially set up much larger than those obtained from standard laboratory oedometer testing. However, as loading increases, a reduction to the coefficient of consolidation values was carried out since the permeability dropped after the soil threshold value has been passed.

Figures 7 and 8 show the comparisons of excess pore water pressure responses at centre line for depth of 7.5 m and 13 m between the back calculation analysis and the recorded data in the field, respectively. It can be observed that the excess pore water pressures highly generated about 126 days after installation of the piezometers. This occured when the height of the embankment passed 2.4 m.

The excess pore water pressures predicted by the proposed method were slightly different from the recorded data in the field. However, there were similar patterns between the two lines. The Skempton's parameters were found to vary between -0.27 to 0.65 [9].

CONCLUSIONS

This study presents the development of the finite difference approach applied to consolidation analysis of embankments. To model the embankment problems as real as possible, the finite difference approach has been performed in two dimensional directions.

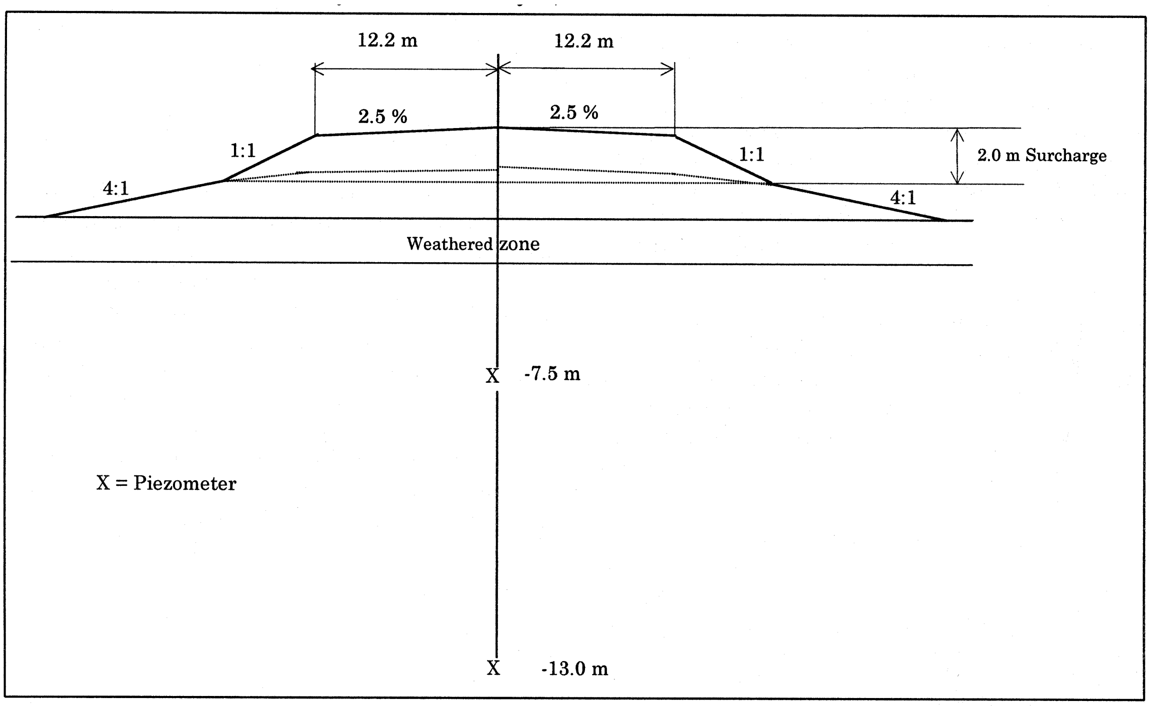

Figure 6. Embankment Test Section at Sta 37+045

Table 2. Soil Properties below the trial embankment at Sta 37+045

	Weathered zone layer	Soft clay layer	Desiccated layer	Soft clay layer	Desiccated layer	Soft clay layer
Depth (m)	3.0	5.0	3.5	5.0	4.0	9.5
γ (kN/m³)	16.0	12.5	12.5	13.0	16.0	13.0
E (kN/m²)	3300.0	2400.0	3500.0	2800.0	3100.0	2600.0
C_v (m²/yr)	75.0	475.0	125.0	125.0	425.0	425.0
C_h (m²/yr)	225.0	1425.0	375.0	375.0	1275.0	1275.0
$P_{c'}$ (kN/m²)	315.0	130.0	150.0	130.0	315.0	260.0

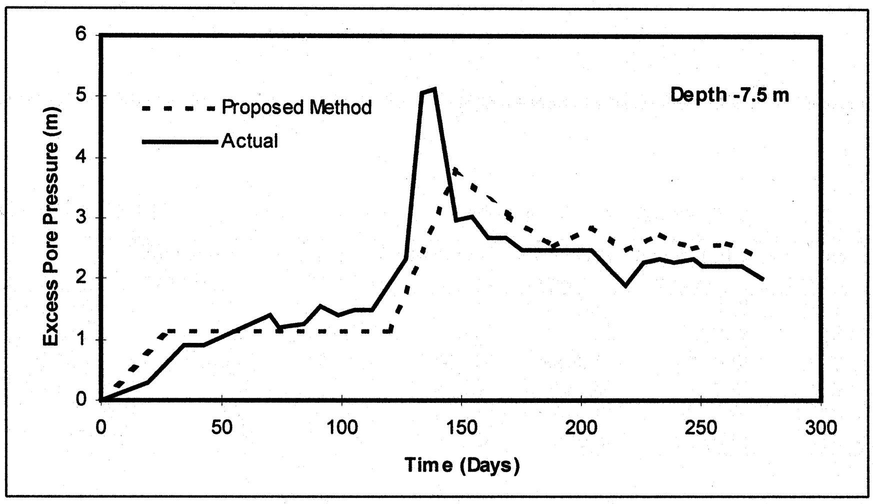

Figure 7. Comparison of Excess Pore Water Pressure Response fo Depth 0f -7.5 m at Centre Line

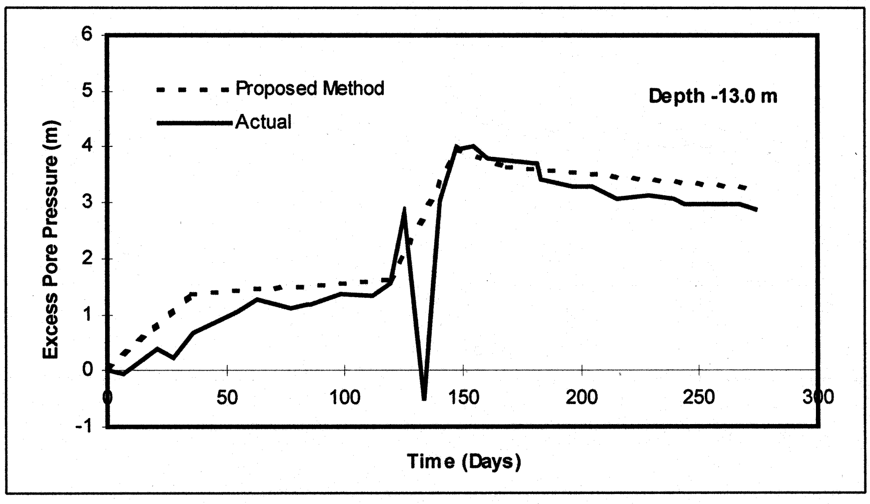

Figure 8. Comparison of Excess Pore Water Pressure Response fo Depth 0f -13.0 m at Centre Line

According to the numerical results of this study, it can be concluded that:

a) The finite difference method is easy to be implemented to solve consolidation problems of embankments.
b) The alternating direction implicit (ADI) finite difference method is a very good method. It is convergent and unconditionally stable during the calculation process.
c) The Skempton's parameter *A* should be considered in analysing the behaviour of excess pore water pressure dissipation. The Skempton's parameters *A* were found to vary between -0.27 to 0.65 for the case of Panci toll highway.
d) For analysing the dissipation behaviour of the excess pore water pressure, C_v and C_h values need to be set up larger than those

obtained from the standard laboratory oedometer testing for the early stage. As loading increases, a reduction to the initial coefficient of consolidation values may be carried out since the permeability drops after the soil threshold value is passed.

e) The excess pore water pressures found between the recorded data and the results of the back calculation analysis were in a good agreement.

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