COMMUTING TRACES OF SYMMETRIC LEFT BICENTRALIZER ON SEMIPRIME RINGS

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Abstract : *Let R be an associative ring. The purpose of this paper is to study the investigate identities satisfied by symmetric left Bicentralizer on a semiprime rings which make the trace of such mappings is commuting on R.*

Key words and phrases: *Semiprime rings, symmetric left Bicentralizer, trace of biadditive mappings, commuting mapping.*

INTRODUCTION

Throughout R will represent an associative ring with center $Z(R)$. A ring R is semiprime if $aRa = \{0\}$ implies $a=0$. An additive mapping $S: R \longrightarrow R$ is called a left (right) centralizer if $S(xy)= S(x)y$ **(**$S(xy) = x S(y)$**)**, for all $x,y \in R$. A mapping $B(. , .): R \times R$ R is called symmetric if $B(x, y) = B(y, x)$ for all pairs $x,y \in R$.

A mapping $f{:}R \longrightarrow R$ defined by $f(x) = B(x , x)$, where $B(. , .): RxR \longrightarrow R$ is a symmetric mapping will be called the trace of B. It is obvious that in case $B(. , .): RxR \longrightarrow R$ is a symmetric mapping which is also biadditive, the trace of B satisfies $f(x+y)= f(x) +2B(x,y) + f(y)$.

A symmetric biadditive mapping $F(. , .){:}RRR$ is called a symmetric left Bicentralizer if for any fixed y, the map $x \longrightarrow F(x, y)$ is a left centralizer on R. The commutator $xy - yx$ will be written as $[x, y]$. Note that $[xu , y]= [x , y]u + x [u , y]$ and $[x, yv]= [x , y]v + y [x , v]$, for all $x,y,u,v \in R$.

A mapping $L{:}\ R\ x\ R$ is said to be centralizing on R if $[L(x), x]$ $Z(R)$, for all x R. In special case when $[L(x), x] =0$, for all x R, the mapping L is called commuting on R.

It seems that the first results on commuting mappings which are not additive was given by Vokman [3]. He prove that if d is a derivation of a prime ring R of characteristic different from 2, such that the mapping $q(x) = [d(x), x]$ is commuting then $q=0$, that is d is commuting. In [7] Bresar generalized this result by showing that the same conclusion holds for any additive mapping. Bresar in [9] describe all commuting traces of biadditive mapping on certain prime rings. He prove that if the characteristic of R different from 2, and R does not satisfy S_4, then every such mapping (say q) is of the form $q(x) = \lambda x^2 + \mu(x)\, x + \nu(x)$ for all $x \in R$, where λ C (the Extended Centroid of R), and : $R \longrightarrow C$ is an additive mapping.

In this paper we generalized some results givens in [5], [4], and [6] by introduce and apply the symmetric left **Bi**centralizer mappings on these results.

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Main results

Theorem 1: Let R be a non commutative semiprime ring of characteristic different from 2, and $F, G: RR R$ are symmetric left Bicentralizers. Suppose that $[F(x, y), G(x, y)] \circ F(x, y) = 0$, for all $x,y \in R$, then $[f(x), g(y)] f(x) = 0$, and $f(x)[f(x), g(y)] = 0$, where f, g are the trace of F, G respectively .

Proof: We have :

$[F(x, y), G(x, y)] F(x, y) + F(x, y) [F(x, y), G(x, y)] = 0$, for all $x,y \in R$. (1)

Replace x by $x+y$ in (1), and apply (1) on the obtained relation gives:

$[F(x, y), g(y)] F(x, y) + [f(y), G(x, y)] F(x, y) + [f(y), g(y)] F(x, y) + [F(x, y), G(x, y)] f(y) + [F(x, y), g(y)] f(y) + [f(y), G(x, y)] f(y) + F(x, y) [F(x, y), g(y)] + F(x, y)[f(y), G(x, y)] + F(x, y)[f(y), g(y)] + f(y)[f(y), G(x, y)] + f(y)[F(x, y), g(y)] + f(y)[F(x, y), G(x, y)] = 0$, for all $x,y \in R$.

Replace x by $-x$ in above relation, and comparing the relation so obtained with it, further using the fact that char$R \neq 2$, we get :

$[f(y), g(y)]F(x, y) + F(x, y) [f(y), g(y)] + [F(x, y), g(y)] f(y) + f(y)[F(x, y), g(y)] + [f(y), G(x, y)] f(y) + f(y)[f(y), G(x, y)] = 0$, for $x,y \in R$. (2)

Replace x by yx in (2), and apply (1) on the obtained relation, we obtain:

$f(y) x[f(y), g(y)] + [f(y), g(y)] x f(y) + f(y) [x, g(y)] + f(y)^2[f(y), g(y)] + [f(y), g(y)]x f(y) + g(y)[f(y), x] f(y) + f(y) [f(y), g(y)] x + f(y)g(y) [f(y), g(y)] = 0$, for all $x,y \in R$. (3)

Putting $xf(y)$ instead of x in (3), we have:

$f(y)x f(y)[f(y), g(y)] + [f(y), g(y)] x f(y)^2 + f(y)[x, g(y)] f(y)^2 + f(y) x [f(y), g(y)] f(y) + f(y)^2[x, g(y)] f(y) + f(y)^2 x [f(y), g(y)] + [f(y), g(y)] x f(y)^2 + g(y) x [f(y), x] f(y)^2 + f(y)[f(y), g(y)] x f(y) + f(y)g(y) [f(y), x)] f(y) = 0$, for all $x,y \in R$. (4)

Applying (3) on (4), we get:

$f(y) x f(y) [f(y), g(y)] + f(y)^2 x [f(y), g(y)] = 0$, for all $x,y \in R$.

Replace x by y and y by x in above relation, we see:

$f(x) y f(x) [f(x), g(x)] + f(x)^2 y [f(x), g(x)] = 0$, for all $x,y \in R$. (5)

Now putting in (5) g(x)y instead of y :

$f(x) g(x) y f(x) [f(x), g(x)] + f(x)^2 g(x) y [f(x), g(x)] = 0$, for all $x,y \in R$ (6)

Left multiplication of (5) by $g(x)$, then subtracting the relation so obtained from (6) leads to:

$[f(x), g(x)] y f(x)[f(x), g(x)] + [f(x)^2, g(x)] y [f(x), g(x)] = 0$, for all $x,y \in R$. (7)

But $[f(x)^2, g(x)] = [f(x), g(x)] f(x) + f(x) [f(x), g(x)] = 0$, therefore (7) reduces to:

$[f(x), g(x)] y f(x) [f(x), g(x)] = 0$, for all $x,y \in R$.

Left multiplication above relation by $f(x)$ yields:

$f(x)[f(x), g(x)] y f(x) [f(x), g(x)] = 0$, for all $x,y \in R$.

Using the semiprimeness of R we conclude that:

$f(x)[f(x), g(x)] = 0$, for all $x,y \in R$. (8)

Now, as a special case of (1) when $y = x$, and using (8), we arrive to:

$[f(x), g(x)] f(x) = 0$, for all $x,y \in R$. ■

Theorem 2: Let R be a 2-torsion free semiprime ring, and $F, G: R\times R \longrightarrow R$ are symmetric left Bicentralizers. Suppose $[[f(x), g(x)], f(x)] = 0$, for all $x,y \in R$, then $[f(x), g(y)]f(x) = 0$, and $f(x)[f(x), g(y)] = 0$, where f, g are the trace of F,G respectively.

Proof: We have:

$$[[f(x), g(x)], f(x)] = 0, \text{ for all } x,y \in R. \quad (1)$$

Linearization of (1) gives:

$$[[f(x), g(x)], f(y)] + [[f(x), g(x)], 2F(x, y)] + [[f(x), 2G(x, y)], f(x)] + [[f(x), 2G(x, y)], f(y)] + [[f(x), 2G(x, y)], 2F(x, y)] + [[f(x), g(y)], f(x)] + [[f(x), g(y)], f(y)] + [[f(x), g(y)], 2F(x, y)] + [[f(y), g(x)], f(y)] + [[f(y), g(x)], 2F(x, y)] + [[f(y), g(x)], f(x)] + [[f(y), 2G(x, y)], f(x)] + [[f(y), 2G(x, y)], f(y)] + [[f(y), 2G(x, y)], 2F(x, y)] + [[f(y), g(y)], f(x)] + [[f(y), g(y)], 2F(x, y)] + [[2F(x, y), g(x)], f(y)] + [[2F(x, y), g(x)], 2F(x, y)] + [[2F(x, y), g(x)], f(x)] + [[2F(x, y), 2G(x, y)], f(x)] + [[2F(x, y), 2G(x, y)], f(y)] + [[2F(x, y), 2G(x, y)], 2F(x, y)] + [[2F(x, y), g(y)], f(x)] + [[2F(x, y), g(y)], 2F(x, y)] + [[2F(x, y), g(y)], f(y)] = 0, \text{ for all } x,y \in R.$$

Replacing x by $-x$, and comparing the relation so obtained with the above, we get after using the fact that R is a 2-torsion free:

$$[[f(x), g(x)], F(x, y)] + [[f(x), G(x, y)], f(x)] + [[f(x), G(x, y)], f(y)] + [[f(x), g(y)], F(x, y)] + [[f(y), g(x)], F(x, y)] + [[f(y), G(x, y)], f(x)] + [[f(y), G(x, y)], f(y)] + [[f(y), g(y)], F(x, y)] + [[F(x, y), g(x)], f(y)] + [[F(x, y), g(x)], f(x)] + [[F(x, y), g(y)], f(x)] + [[F(x, y), g(y)], f(y)] + [[2F(x, y), 2G(x, y)], F(x, y)] = 0, \text{ for all } x,y \in R. \quad (2)$$

Substituting $2x$ instead of x in (2), comparing the relation so obtained with (2), and using the fact that R is a 2-torsion free, we see:

$$[[f(x), g(x)], F(x, y)] + [[f(x), G(x, y)], f(x)] + [[F(x, y), g(x)], f(x)] = 0, \text{ for all } x,y \in R. \quad (3)$$

Replace y by xy in (3), and apply (1) on the relation so obtained, we arrive:

$$f(x)[[f(x), g(x)], y] + 3[f(x), g(x)][y, f(x)] + g(x)[[f(x), y], f(x)] + f(x)[[y, g(x)], f(x)] = 0, \text{ for all } x,y \in R \quad (4)$$

Putting $yf(x)$ for y in (4), and using the identity $[xy, z] = x[y, z] + [x, z]y$, we have :

$$f(x)[[f(x), g(x)], y] f(x) + f(x)y[[f(x), g(x)], f(x)] + 3[f(x), g(x)][y, f(x)] f(x) + g(x)[[f(x), y], f(x)] f(x) + f(x)[[y, f(x)][f(x), g(x)] + f(x)y[[f(x), g(x)], f(x)] + f(x)[[y, g(x)], f(x)] f(x) = 0, \text{ for all } x,y \in R. \quad (5)$$

In view of (1) and (4), the relation (5) reduces to:

$$f(x)\, y\, f(x)[[f(x), g(x)] = f(x)^2\, y\, [[f(x), g(x)] = 0, \text{ for all } x,y \in R. \quad (6)$$

Replace y by $g(x)y$ in (6) yields:

$$f(x)g(x)\, y\, f(x)[[f(x), g(x)] = f(x)^2\, g(x)y\, [[f(x), g(x)] = 0, \text{ for all } x,y \in R. \quad (7)$$

Left multiplication of (6) by $g(x)$, then subtracting the relation so obtained from (7), we obtain:

$$[f(x), g(x)]\, y\, f(x)[f(x), g(x)] = [f(x)^2, g(x)]\, [f(x), g(x)]$$

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$= ([f(x), g(x)]f(x)+f(x)[f(x), g(x)])[f(x), g(x)]$

According to the requirement of this Theorem, we can replace $[f(x), g(x)]f(x)$ by $f(x)[f(x), g(x)]$ in the right hand side, which gives :

$[f(x), g(x)]y f(x)[f(x), g(x)] = 2 f(x)[f(x), g(x)]y [[f(x), g(x)]$

Left multiplication of the above relation f(x) gives:

$f(x)[f(x), g(x)]y f(x) [f(x), g(x)] =2 f(x)^2[f(x), g(x)]y [[f(x), g(x)]$ (8)

On the other hand, putting $[f(x), g(x)]y$ for y, and comparing the relation so obtained with (8), we arrive to:

$f(x)[f(x), g(x)]y f(x)[f(x), g(x)] =0,$ for all $x,y \in R$.

Hence, by semiprimeness of R, it follows:

$f(x)[f(x), g(x)] = 0,$ for all $x \in R$.

Finally, from the last equation and our assumption (1), we have:

$[f(x), g(x)]f(x) = 0,$ for all $x \in R$. ■

Theorem 3 : Let R be a non-commutative semiprime ring of characteristic different from 2, and $F,G: R\times R \longrightarrow R$ are symmetric left Bicentralizers. Suppose $f(x)[f(x), g(x)] = 0,$ and $[f(x), g(x)]f(x) = 0,$ for all x R, then $[f(x), g(x)] =0,$ for all $x \in R$, where f, g are the trace of F, G respectively.

Proof: We have:

$f(x) [f(x), g(x)] =0,$ for all $x,y \in R$. (1)

Now, using similar techniques as used to get (3) from (1) in Theorem (2), and the fact that char$R2$, we arrive at :

$f(x)[f(x), G(x, y)]+ f(x)[F(x, y), g(x)]+ F(x, y)[f(x), g(x)] = 0,$ for all $x,y \in R$. (2)

Putting in (2) xy for y, and apply the identity $[xy, z]= x[y, z]+[x, z]y$ on the relation so obtained, we get :

$f(x)[f(x), g(x)]y + f(x) g(x)[f(x), y]+ f(x)^2[y, g(x)] + f(x)[f(x), g(x)]y + f(x)y[f(x), g(x)]=0,$ for all $x,y \in R$.

In view of (1), above relation reduces to:

$f(x) g(x)[f(x), y]+ f(x)^2[y, g(x)]+ f(x) y[f(x), g(x)] =0,$ for all $x,y \in R$.

The above relation can be written as:

$f(x)y[f(x), g(x)]+ f(x)^2 y g(x) - f(x) g(x) y f(x) + f(x)[f(x), g(x)]y =0.$

According to (1), above relation reduce to:

$f(x)y[f(x), g(x)] + f(x)^2 y g(x) - f(x) g(x) y f(x)= 0,$ for all $x,y \in R$. (3)

Now, substitution $g(x)y$ instead of y in (3) gives:

$f(x) g(x) y [f(x), g(x)] + f(x)^2 g(x) y g(x) - f(x) g(x)^2 y f(x)= 0.$ (4)

Left multiplication of (3) by $g(x)$ and subtracting the relation so obtained from (4) yields:

$[f(x), g(x)]y [f(x), g(x)] + [f(x)^2, g(x)]y g(x) + [g(x), f(x)]g(x) y f(x)= 0.$

But $[f(x)^2, g(x)] = [f(x), g(x)]f(x)+f(x)[f(x), g(x)]=0,$ then above relation reduces to:

$[f(x), g(x)]y [f(x), g(x)] + [g(x), f(x)]g(x) y f(x)= 0$, for all $x,y \in R$. (5)

Putting $y f(x) z$ for y in (5) gives:

$[f(x), g(x)]y f(x) z [f(x), g(x)] + [g(x), f(x)] g(x) y f(x) z f(x)= 0.$ (6)

Right multiplication of (5) by $z f(x)$, and subtracting (6) from the relation so obtained leads to:

$[f(x), g(x)]y \mu(x,z) = 0,$ for all $x,z \in R$. (7)

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Where $\mu(x, z) = [f(x), g(x)]\, z\, f(x) - f(x)\, z[f(x), g(x)]$

Again replace y by $z f(x)y$ in (7), leads to :

$[f(x),g(x)]\, z\, f(x)\, y\, \mu(x,z)=0$, for all $x,z \in R$. (8)

Left multiplication of (7) by $f(x)z$, then subtracting the relation so obtained from (8) yields:

$\mu(x, z)\, y\, (x,z) = 0$, for all $x,y,z \in R$.

Using the semiprimeness of R, we conclude that:

$[f(x), g(x)]\, z\, f(x) = f(x)\, z[f(x), g(x)]$ (9)

Now putting $g(x)y$ for z in (9) implies that:

$[f(x), g(x)]\, g(x)y\, f(x) = f(x)\, g(x)y[f(x), g(x)]$ (10)

In view of (10) the relation (5) at once yields that:

$[f(x), g(x)]\, y\, [f(x), g(x)] - f(x)\, g(x)y[f(x), g(x)]=0.$

This can be reduces to:

$g(x)f(x)\, y[f(x), g(x)]= 0$, for all $x,y \in R$. (11)

Replacing y by $g(x)y$ in (11) gives :

$g(x)\, f(x)\, g(x)\, y\, [f(x), g(x)] = 0$, for all $x,y \in R$.

Also left multiplication of (11) by $g(x)$ and subtracting the relation so obtained from the above, we get:

$g(x)[f(x), g(x)]\, y\, [f(x), g(x)]= 0$, for all $x,y \in R$. (12)

Putting $yg(x)$ for y in (12) and using the semiprimeness property of R, we conclude:

$g(x)\, [f(x), g(x)] =0$, for all $x,y \in R$. (13)

The substitution $y\, g(x)$ for z in (9) gives because of (13) :

$[f(x), g(x)]\, y\, g(x)\, f(x) = 0$, for all $x,y \in R$. (14)

For the second condition $[f(x), g(x)]\, f(x) =0$, using the similar techniques as used to get (3) in theorem (2), and the fact that char$R$ 2, one obtains:

$[f(x), G(x, y)]\, f(x)+[F(x, y), g(x)]\, f(x)+[f(x), g(x)]\, F(x, y) = 0$, for all $x,y \in R$.

Replace y by xy in above relation leads to:

$2[f(x), g(x)]\, y\, f(x)+ g(x)[f(x), y]\, f(x) + f(x)[y, g(x)]\, f(x) =0$, which can be written as :

$[f(x), g(x)]\, y\, f(x) + f(x)\, y\, g(x)\, f(x) - f(x)\, y\, f(x)^2 =0$, for all $x,y \in R$.

In view of (9) the above relation at once yields that:

$f(x)\, y[f(x), g(x)]+ f(x)\, y\, g(x)\, f(x) - g(x)\, y\, f(x)^2=0$, for all $x,y \in R$.

This can reduce to $f(x)\, y\, f(x)\, g(x) - g(x)\, y\, f(x)^2 =0$, and consequently :

$f(x)\, y\, f(x)\, g(x) = g(x)\, y\, f(x)^2$ (15)

Putting $g(x)y$ for y in (15) leads to :

$f(x)\, g(x)\, y\, f(x)\, g(x) = g(x)^2\, y\, f(x)^2$ (16)

Left multiplication of (15) by $g(x)$, and subtracting the relation so obtained from (16) leads to:

$[f(x),g(x)]y f(x)\, g(x)=0$, for all $x,y \in R$. (17)

Combining (14) with (17), we get:

$[f(x), g(x)]\, y\, [f(x), g(x)] = 0$, for all $x,y \in R$.

By semiprimeness property of R yields:

$[f(x), g(x)] = 0$, for all $x,y \in R$.

Now, Theorems (1), (2) and (3) leads to the following results.

Corollary A: Let R be a non-commutative semiprime ring of characteristic different from 2, and $F,G: R \times R \longrightarrow R$ are symmetric left Bicentralizers. Suppose that $[F(x, y), G(x, y)]\, o\, F(x, y) =0$, for all $x,y \in R$, then $[f(x), g(x)] =0$, for all $x \in R$, where f, g are the trace of F, G respectively .

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Corollary B: Let R be a 2-torsion free semiprime ring, and $F: R \times R \longrightarrow R$ be symmetric left Bicentralizer. Suppose that $[F(x, y), x]\ o\ x = 0,$ for all $x \in R$, then the trace of F (say f) is commuting on R.

Corollary C: Let R be a 2-torsion free semiprime ring, and $F: R \times R \longrightarrow R$ be symmetric left Bicentralizer. Suppose that $[[f(x), x], x] = 0$, for all $x \in R$, where f is the trace of F, then f is commuting on R.

Corollary D: Let R be a 2-torsion free semiprime ring, and $F: R \times R \longrightarrow R$ be symmetric left Bicentralizer. Suppose that $[[f(x), x], f(x)] = 0,$ for all $x \in R$, where f is the trace of F, then f is commuting on R.

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