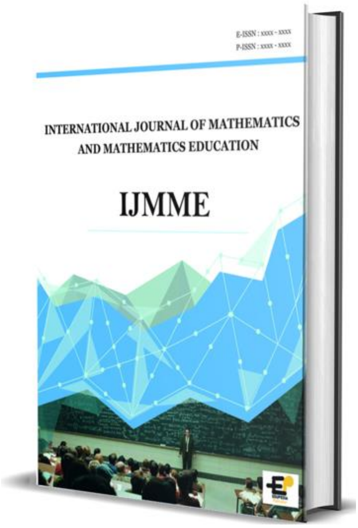

https://journals.eduped.org/index.php/IJMME

Mathematical Induction, Transfinite Induction, and Induction Over The Continuum

Denik Agustito*

Universitas Sarjanawiyata Tamansiswa, Indonesia

Sukiyanto

Universitas Sarjanawiyata Tamansiswa, Indonesia

Krida Singgih Kuncoro

Universitas Sarjanawiyata Tamansiswa, Indonesia

To cite this article:

Agustito, *et. el.*, (2023). Mathematical Induction, Transfinite Induction, and Induction Over The Continuum. *International Journal of Mathematics and Mathematics Education (IJMME), 1*(2), 180-191 https://doi.org/10.56855/ijmme.v1i02.385

IJMME International Journal of Mathematics and Mathematics Education
June 2023, Vol. 01, No. 02, 180-191 doi: 10.56855/ijmme.v1i02.385

Mathematical Induction, Transfinite Induction, and Induction Over The Continuum

Denik Agustito[1*], Sukiyanto[2], Krida Singgih Kuncoro[3]

Article Info

Article History

Received:
May 26, 2023
Accepted:
June 01, 2023

Keywords

Continuum
Induction
Mathematics
Ordinal
Transfinite

Abstract

This article examines three types of induction methods in mathematics: mathematical induction, transfinite induction, and induction over the continuum. If a statement holds true for all natural numbers, it is proven using mathematical induction. If a statement holds true for all ordinal numbers, it is proven using transfinite induction. Since induction over the continuum cannot be applied to a statement, when something is said to be proven true for every point in $[a, b)$, the proof is done using induction over the continuum.

Introduction

In conducting mathematical activities, reasoning in mathematics plays a fundamental role as the foundation or method for performing mathematics. The commonly known methods are deductive reasoning and inductive reasoning. Deductive reasoning is a process where specific facts are derived from a generally known truth, while inductive reasoning in mathematics is based on the principle of induction. The principle of induction asserts that a universal truth is demonstrated by showing that if a specific case is true and subsequently a sufficient number of cases are also true (Singh, Yadav:2017). An example of inductive reasoning in mathematics is prominently seen in mathematical induction. Mathematical induction states that a statement P holds true for every natural number $n \in \mathbb{N}$ if it satisfies the following properties:

(i). The statement P is true for $n = 1$; usually denoted as $P(1)$ is true.

(ii). If the statement P is true for $n = k$, then the statement P is also true for $n = k + 1$; typically written as "if $P(k)$ is true, then $P(k + 1)$ is also true."

Mathematical induction turns out to be inadequate as an inductive method used in mathematics for certain cases related to the concept of true infinity (actual infinity), such as the properties of transfinite arithmetic in set theory and the Heine-Borel property in real line topology. To overcome this issue, an inductive method in mathematics that goes beyond mathematical induction needs to be developed in order to obtain valid truths about many mathematical problems related to true infinity. The inductive methods to be discussed in this article are transfinite induction (Goldrei: 1996) and induction over the continuum (Kalantari: 2007).

Method

The method used in this research is a literature review. The first step involves collecting relevant literature such as books and papers on mathematical induction, transfinite induction, and induction over the continuum. Next, the ideas related to induction are organized in a sequence, starting from the most fundamental concepts such as mathematical induction along with examples, followed by transfinite induction with examples, and finally induction over the continuum with examples.

Discussion

This section will begin with mathematical induction. According to (Hungerford: 1974), the principle of mathematical induction is as follows:

The principle of Mathematical Induction. *If S is a subset of the set of natural numbers denoted by $\mathbb{N}$ and satisfies the property $0 \in S$, and one of the following statements holds true:*

(i). $n \in S \implies n + 1 \in S$ *for all* $n \in N$, *or*

(ii). $m \in S$ *for all* $0 \leq m < n \implies n \in S$ *for all* $n \in N$.

then $S = \mathbb{N}$.

The principle of mathematical induction operates within set theory. This means that when proving that $S = \mathbb{N}$, the set S must satisfy certain properties. Various literature describes the principle of mathematical induction, with Hungerford providing a more specific formulation, as follows:

Mathematical Induction. *If S is a subset of the set of natural numbers denoted by $\mathbb{N}$, which satisfies the following properties:*

(i). $0 \in S$.

(ii). $n \in S \implies n + 1 \in S$ *for all* $n \in \mathbb{N}$.

then $S = \mathbb{N}$.

In Hungerford's formulation of mathematical induction, the set of all natural numbers denoted by $\mathbb{N}$ is not initiated from the number 1, but rather from the number 0. However, there is a difference in mathematical induction in logic, where the set of all natural numbers denoted by $\mathbb{N}$ starts from the number 1. The mathematical induction in the logical version can be expressed as follows (Herstein: 1986):

Mathematical Induction. *Statement P holds true for every natural number $n \in \mathbb{N}$ if it satisfies the following properties:*

(i). *Statement P is true for $n = 1$; usually denoted as $P(1)$ being true.*

(ii). *If statement P is true for $n = k$, then statement P is also true for $n = k + 1$; typically written as "if $P(k)$ is true, then $P(k + 1)$ is also true."*

Then, an example of the usage of logical version mathematical induction is as follows:

Example 1.1. Prove that for every natural number n, the following equality holds:

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$

Proof:

Bukti:

Let's write the statement $P(n): 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for natural number n.

(i). Prove that $P(n)$ holds true for $n = 1$.

Clearly, $P(1): 1 = \frac{1(1+1)}{2}$ is a true statement.

Therefore $P(n)$ holds true for $n = 1$.

(ii). Prove that if $P(n)$ holds true for $n = k$, then $P(n)$ holds true for $n = k + 1$.

Assume that $P(n)$ holds true for $n = k$.

Clearly, $P(k): 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$ is a true statement.

Now, consider
$$\begin{aligned}(1 + 2 + 3 + \cdots + k) + (k + 1) &= \frac{k(k+1)}{2} + (k + 1)\\ &= \frac{k(k+1)+2(k+1)}{2}\\ &= \frac{(k+1)(k+2)}{2}\\ &= \frac{(k+1)\big((k+1)+1\big)}{2}\end{aligned}$$

Therefore $P(k + 1): 1 + 2 + 3 + \cdots + (k + 1) = \frac{(k+1)((k+1)+1)}{2}$ is a true statement.

Hence, the statement $P(n): 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for natural number n.

Next, we will examine transfinite induction as one of the accepted inductive methods in mathematics, applied to a statement that holds true for all ordinal numbers (both finite or infinite ordinal numbers, as well as transfinite ordinal numbers). Before introducing what transfinite induction is, we first introduce the concept of ordinal numbers and some of their properties, referring to (Jech: 2003).

Definition 1.2. *Let P be a set and < be a relation on P. The relation < is said to be a partial order if it satisfies the following properties:*

(i). *$p \nless p$ for every $p \in P$ (the relation < is anti-reflexive).*

(ii). *If $p < q$ and $q < r$ then $p < r$ for every $p, q, r \in P$ (the relation < is transitive).*

And the pair $(P, <)$ is called a partially ordered set.

Next, the concept of special elements in a partially ordered set $(P, <)$ will be introduced.

Definition 1.3. *Given a partially ordered set* $(P, <)$ and X *a non-empty subset of* P, *and* $a \in P$, *then*:

(i). *a is called a maximal element of X if* $a \in X$ *and* $\forall x \in X, a \nless x$.

(ii). *a is called a minimal element of X if* $a \in X$ *and* $\forall x \in X, x \nless a$.

(iii). *a is called the greatest element of X if* $a \in X$ *and* $\forall x \in X, x \leq a$.

(iv). *a is called the least element of X if* $a \in X$ *and* $\forall x \in X, a \leq x$.

(v). *a is called an upper bound of X if* $\forall x \in X, x \leq a$.

(vi). *a is called a lower bound of X if* $\forall x \in X, a \leq x$.

(vii). *a is called the supremum of X if a is the smallest element among all upper bounds of X.*

(viii). *a is called the infimum of X if a is the largest element among all lower bounds of X.*

In a partially ordered set $(P, <)$, the ordering relation $<$ is said to be linearly ordered or totally ordered if $\forall p, q \in P$ one of the following holds $p < q$ or $p = q$ or $p > q$.

Definition 1.4. *If the ordering relation* $<$ *on a partially ordered set* $(P, <)$ *is linearly ordered and every non-empty subset of* P *has a least element, then the set* P *is said to be well-ordered.*

Next, the idea to construct ordinal numbers is by introducing a relation on them; meaning that given two ordinal numbers α and β, the partially ordered relation $<$ is defined as follows:

$$\alpha < \beta \overset{\text{def}}{=} \alpha \in \beta$$

from this concept, an understanding of transitive sets is obtained.

Definition 1.5. *A set* T *is said to be transitive if every element of* T *is a subset of* T.

Definition 1.6. *A set is called an ordinal number if it is both transitive and well-ordered.*

Usually, ordinal numbers are denoted by $\alpha, \beta, \gamma, \ldots$ and the class of all ordinal numbers is denoted by $\mathcal{O}$. From the concept of ordinal numbers, further constructions of ordinal numbers are made, such as *finite ordinals* and *transfinite ordinals*.

(*). Constructing finite ordinal numbers.

Define the finite ordinal number 0 as $0 \overset{\text{def}}{=} \emptyset$.

Next, the concept of the *successor* of a set X will be introduced as follows:

$$X^+ \overset{\text{def}}{=} X \bigcup \{X\}$$

The set X^+ will be called the successor of the set X.

Through the concept of the *successor* of a set, finite ordinal numbers will be defined as follows:

$$1 \stackrel{\text{def}}{=} 0^+ = \emptyset^+ = \emptyset \bigcup \{\emptyset\} = \{\emptyset\} = \{0\}$$

$$2 = 1^+ = \{0,1\}$$

$$3 = 2^+ = \{0,1,2\}$$

$$\vdots$$

So, all natural numbers can be viewed as finite ordinal numbers.

(**). Constructing transfinite ordinal numbers.
Now let's consider the pattern of finite ordinal numbers:

$$1 = \{0\}$$

$$2 = \{0,1\}$$

$$3 = \{0,1,2\}$$

$$\vdots$$

$$X = \{0,1,2,3,\dots\}$$

The question is, who is X? In set theory, X represents the first transfinite ordinal number and is denoted as ω. So $\omega = \{0,1,2,\dots\}$ is the first transfinite ordinal number. Since ω is the first transfinite ordinal number, the notion of *successor* of a set can be applied to ω as follows:

$$\omega + 1 \stackrel{\text{def}}{=} \omega^+ = \{0,1,2,\dots,\omega\}$$

$$\omega + 2 = (\omega + 1)^+ = \{0,1,2,\dots,\omega,\omega + 1\}$$

$$\omega + 3 = (\omega + 2)^+ = \{0,1,2,\dots,\omega,\omega + 1,\omega + 2\}$$

$$\vdots$$

From this construction, we obtain the class of all ordinal numbers, both finite and transfinite, as follows:

$$\mathcal{O} = \{0,1,\dots,\omega,\omega + 1,\dots,\omega.2,\omega.2 + 1,\dots,\omega^\omega,\omega^\omega + 1,\dots\}$$

(***). Classification of ordinal numbers.

Ordinal numbers are classified into two categories as follows:

(i). *Successor* ordinal numbers. An ordinal number α is called a successor ordinal if there exists an ordinal number β such that $\alpha = \beta^+ = \beta + 1$.
Examples of *successor* ordinal numbers include all finite ordinal numbers, as well as $\omega + 1, \omega + 2. \omega + 3, \ldots$ and so on.

(ii). *Limit* ordinal numbers. An ordinal number α is called a *limit* ordinal if $\alpha = \sup\{\beta | \beta < \alpha\}$.
Examples of *limit* ordinal numbers include $\omega, \omega. 2, \ldots, \omega^\omega, \ldots$ and so on.

Just like in mathematical induction, which provides a condition for a statement to hold true for all natural numbers, a similar concept is introduced to examine the validity of a statement that applies to all ordinals. This concept is known as transfinite induction. According to (Andre: 2018), transfinite induction is defined as follows:

Transfinite Induction. *Let $\{x_\alpha | \alpha \in \mathcal{O}\}$ be a class indexed by ordinals, and let P be a property of elements. Suppose $P(\alpha)$:"element x_α satisfies property P" and for every $\alpha \in \mathcal{O}$ the following holds:*

$$P(\beta) \textit{ is true } \forall \beta < \alpha \Longrightarrow P(\alpha) \textit{ is true}$$

then $P(\beta)$ is true for all ordinal $\beta \in \mathcal{O}$.

The definition of transfinite induction above can be simplified as follows:

Transfinite Induction. *A statement P is true for all ordinal numbers $\alpha \in \mathcal{O}$ if it satisfies the following properties:*

(i). *The statement P is true for $\alpha = 0$; usually denoted as $P(0)$ being true.*

(ii). *If α is a successor ordinal $(\exists \beta \in \mathcal{O}[\alpha = \beta^+ = \beta + 1]$ and P is true for β then P is also true for $\alpha = \beta^+ = \beta + 1$: It is usually denoted as if $P(\beta)$ is true, then $P(\alpha) = P(\beta^+) = P(\beta + 1)$ is also true.*

(iii). *If α is a limit ordinal $(\alpha = \sup\{\beta | \beta < \alpha\}$ and P is true for all β with $\beta < \alpha$, then P is also true for α; It is usually denoted as if $P(\beta)$ is true for all β with $\beta < \alpha$, then$P(\alpha)$ is also true.*

Some examples of statements that can be proven true using transfinite induction are the properties of transfinite arithmetic. Transfinite arithmetic refers to arithmetic operations involving both finite and transfinite ordinals.

Transfinite Arithmetic.

(i). Addition of ordinal numbers. Given two ordinal numbers, α and β, the addition of these two ordinal numbers is defined as follows:

$$\alpha + \beta = \begin{cases} \alpha & ,\beta = 0 \\ (\alpha + \gamma) + 1 & ,\beta = \gamma^{+} \\ \sup\{\alpha + \gamma | \gamma < \beta\} & ,\beta = \sup\{\gamma | \gamma < \beta\} \end{cases}$$

(ii). Multiplication of ordinal numbers. Given two ordinal numbers, α and β, the multiplication of these two ordinal numbers is defined as follows:

$$\alpha . \beta = \begin{cases} 0 & ,\beta = 0 \\ (\alpha . \gamma) + \alpha & ,\beta = \gamma^{+} \\ \sup\{\alpha . \gamma | \gamma < \beta\} & ,\beta = \sup\{\gamma | \gamma < \beta\} \end{cases}$$

(iii). Exponentiation of ordinal numbers. Given two ordinal numbers, α and β, the exponentiation of these two ordinal numbers is defined as follows:

$$\alpha^{\beta} = \begin{cases} 1 & ,\beta = 0 \\ (\alpha^{\gamma}) . \alpha & ,\beta = \gamma^{+} \\ \sup\{\alpha^{\gamma} | \gamma < \beta\} & ,\beta = \sup\{\gamma | \gamma < \beta\} \end{cases}$$

Some properties of transfinite arithmetic will be proven using transfinite induction, as follows:

Example 1.7. (Associative Property of Ordinal Addition). If three ordinal numbers α, β and γ are given, then

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

Proof:
Consider the statement P defined for all ordinals γ as follows:

$$P(\gamma): "(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)"$$

Transfinite induction will be used to prove that the statement $P(\gamma)$ is true for all ordinals γ.

(i). Case $\gamma = 0$:

Clearly, $(\alpha + \beta) + \gamma = (\alpha + \beta) + 0$

$= \alpha + \beta$

$= \alpha + (\beta + 0)$

$= \alpha + (\beta + \gamma)$

Clearly, $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

Thus, $P(\gamma)$ is true for $\gamma = 0$.

(ii). Case $\gamma = \delta^{+} = \delta + 1$:

Assume that statement P is true for δ; meaning $P(\delta)$ is true.

Since $P(\delta)$ is true, it is clear that $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$.

Clearly, $(\alpha+\beta)+\delta^{+}=(\alpha+\beta)+(\delta+1)$

$=\big((\alpha+\beta)+\delta\big)+1$ ... (based on the definition of addition)

$=\big(\alpha+(\beta+\delta)\big)+1$ ... (based on the hypothesis)

$=\alpha+\big((\beta+\delta)+1\big)$ ... (based on the definition of addition)

$=\alpha+\big(\beta+(\delta+1)\big)$ ... (based on the definition of addition)

$=\alpha+(\beta+\delta^{+})$

Clearly, $(\alpha+\beta)+\delta^{+}=\alpha+(\beta+\delta^{+})$.

Thus, $P(\gamma)$ is true for $\gamma=\delta^{+}=\delta+1$.

(iii). Case $\gamma=\sup\{\delta|\delta<\gamma\}$:

Assume that statement P is true for δ; meaning $P(\delta)$ is true.

Since $P(\delta)$ is true, it is clear that $(\alpha+\beta)+\delta=\alpha+(\beta+\delta)$.

Clearly, $(\alpha+\beta)+\gamma=\sup\{(\alpha+\beta)+\delta|\delta<\gamma\}$ ... (based on the definition of addition)

$=\sup\{\alpha+(\beta+\delta)|\delta<\gamma\}$ ... (based on the hypothesis)

$=\alpha+\sup\{\beta+\delta|\delta<\gamma\}$ ... (property of supremum)

$=\alpha+(\beta+\gamma)$ ... (based on the definition of addition)

Clearly, $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$.

Thus, $P(\gamma)$ holds for $\gamma=\sup\{\delta|\delta<\gamma\}$.

From (i), (ii) and (iii), it is evident that statement $P(\gamma)$ is true for all ordinal numbers γ.
Therefore, for all ordinal numbers α,β and γ, it holds that $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$.

Example 1.8. (Associative Property of Multiplication of Ordinal Numbers). Given three ordinal numbers α,β and γ, we want to prove that

$$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$$

Proof:
Let's consider the statement P defined for all ordinal numbers γ as follows:

$$P(\gamma)\text{: "}(\alpha.\beta).\gamma=\alpha.(\beta.\gamma)\text{"}$$

We will use transfinite induction to prove that the statement $P(\gamma)$ is true for all ordinal numbers γ.

(i). Case $\gamma=0$:

Clearly, $(\alpha.\beta).\gamma=(\alpha.\beta).0$

$=0$

$=\alpha.(\beta.0)$

$=\alpha.(\beta.\gamma)$

Clearly, $(\alpha.\beta).\gamma=\alpha.(\beta.\gamma)$.

Thus, $P(\gamma)$ is true for $\gamma=0$.

(ii). Case $\gamma=\delta^{+}=\delta+1$:

Assume that the statement P is true for δ; meaning $P(\delta)$ is true.

Since $P(\delta)$ is true, it is clear that $(\alpha.\beta).\delta = \alpha.(\beta.\delta)$.

Clearly, $(\alpha.\beta).\delta^{+} = (\alpha.\beta).(\delta + 1)$

$= \big((\alpha.\beta).\delta\big) + (\alpha.\beta)$ ... (based on the definition of multiplication)

$= \big(\alpha.(\beta.\delta)\big) + (\alpha.\beta)$ ... (based on the hypothesis)

$= \alpha.\big((\beta.\delta) + \beta\big)$

$= \alpha.\big(\beta.(\delta + 1)\big)$ ... (based on the definition of multiplication)

$= \alpha.(\beta.\delta^{+})$

Clearly, $(\alpha.\beta).\delta^{+} = \alpha.(\beta.\delta^{+})$.

Thus, $P(\gamma)$ is true for $\gamma = \delta^{+} = \delta + 1$.

(iii). Case $\gamma = \sup\{\delta | \delta < \gamma\}$:

Assume that the statement P is true for δ; meaning $P(\delta)$ is true.

Since $P(\delta)$ is true, it is clear that $(\alpha.\beta).\delta = \alpha.(\beta.\delta)$.

Clearly, $(\alpha.\beta).\gamma = \sup\{(\alpha.\beta).\delta | \delta < \gamma\}$ ... (based on the definition of multiplication)

$= \sup\{\alpha.(\beta.\delta) | \delta < \gamma\}$ ... (based on the hypothesis)

$= \alpha.\sup\{\beta.\delta | \delta < \gamma\}$ ... (property of supremum)

$= \alpha.(\beta.\gamma)$ ... (based on the definition of multiplication)

Clearly, $(\alpha.\beta).\gamma = \alpha.(\beta.\gamma)$.

Thus, $P(\gamma)$ holds for $\gamma = \sup\{\delta | \delta < \gamma\}$.

From (i), (ii) and (iii), it is evident that the statement $P(\gamma)$ is true for all ordinal numbers γ. Therefore, for all ordinal numbers α, β and γ, it holds that $(\alpha.\beta).\gamma = \alpha.(\beta.\gamma)$.

Some properties of transfinite arithmetic, specifically those related to the exponentiation of ordinal numbers, can be proven using transfinite induction. One of these properties states that for any three ordinal numbers α, β and γ, we have $\alpha^{\beta+\gamma} = \alpha^{\beta}.\alpha^{\gamma}$.

■

Another method of induction, apart from mathematical induction and transfinite induction, is the induction over the continuum introduced by Kalantari (2007). While mathematical induction focuses on statements that hold for all natural numbers and transfinite induction focuses on statements that hold for all ordinal numbers, induction over the continuum focuses on statements that hold for numbers within the interval $[a, b) = \{x \in \mathbb{R} | x = a \text{ atau } x < b\}$. The definition of induction over the continuum will now be provided.

Induction over the continuum. *Given two real numbers a, b, where $a < b$ and $S \subseteq [a, b)$ satisfies the following properties:*

(i). $\exists x (x \geq a)$ *and* $[a, x) \subseteq S$.

(ii). $\forall x\big[[a, x) \subseteq S \Longrightarrow \exists y > x([a, y) \subseteq S)\big]$.

then $S = [a, b)$.

Induction over the continuum is not as easily applicable as mathematical induction and transfinite induction for a statement. This method is used in various problems in real analysis, such as continuous functions and uniformly continuous functions (Barttle:2011) and (Royden: 2010), as follows:

Example 1.9. If $f: [a, b] \longrightarrow \mathbb{R}$ is a continuous function, then f is uniformly continuous.
Proof:
Given that $f: [a, b] \longrightarrow \mathbb{R}$ is continuous.
Take any $\varepsilon > 0$.
Define the following set:

$$S = \{t | t \in [a, b), \exists\delta_\varepsilon > 0(\forall u, v \in [a, t), |u - v| < \delta_\varepsilon \Longrightarrow |f(u) - f(v)| < \varepsilon)\}$$

We will prove that $S = [a, b)$ using induction over the continuum.

(i). Choose $x = a$.
Clearly, $x = a \geq a$.
It is evident that f is continuous at a.
It is trivially true that $\exists\delta_\varepsilon > 0(\forall u, v \in [a, a) = \{a\}, |u - v| < \delta_\varepsilon \Longrightarrow |f(u) - f(v)| < \varepsilon)$.
It is clear that $[a, a) = \{a\} \subseteq S$.
Hence $\exists x(x \geq a)$ and $[a, x) \subseteq S$.

(ii). Take any x such that $[a, x) \subseteq S$.
We will prove $\exists y > x, [a, y) \subseteq S$.
Consider x and δ_ε given above.
Since f is continuous at x, it is clear that $\exists\delta_{\left(x,\frac{\varepsilon}{2}\right)} < \min\{x - 1, b - x\}, \forall x', |x - x'| < \delta_{\left(x,\frac{\varepsilon}{2}\right)} \Longrightarrow |f(x) - f(x')| < \frac{\varepsilon}{2}$.
Let's denote $t = x - \frac{1}{2}\delta_{\left(x,\frac{\varepsilon}{2}\right)}$.
Clearly, $t \in S$.
It is evident that $\exists\delta_\varepsilon > 0, \forall u, v \in [a, t], |u - v| < \delta_\varepsilon \Longrightarrow |f(u) - f(v)| < \varepsilon$.
Let's denote $y = x + \frac{1}{2}\delta_{\left(x,\frac{\varepsilon}{2}\right)}$.
We will prove $[a, y) \subseteq S$.

Take any $p \in [a, y)$.
Clearly, $a \leq p < y = x + \frac{1}{2}\delta_{\left(x,\frac{\varepsilon}{2}\right)}$.
Since $\delta_{\left(x,\frac{\varepsilon}{2}\right)} < \min\{x - 1, b - x\}$, jelas $\delta_{\left(x,\frac{\varepsilon}{2}\right)} < b - x$.
Clearly, $\frac{1}{2}\delta_{\left(x,\frac{\varepsilon}{2}\right)} < b - x$.
Therefore, $y = x + \frac{1}{2}\delta_{\left(x,\frac{\varepsilon}{2}\right)} < b$.
Hence, $a \leq p < y < b$.
Therefore, $p \in [a, b)$. ...(i).

Choose $\delta_\varepsilon^* = \min\left\{\delta_\varepsilon, \frac{1}{2}\delta_{\left(x,\frac{\varepsilon}{2}\right)}\right\}$.

Given $\forall u, v \in [a, p], |u - v| < \delta_\varepsilon^* < \delta_\varepsilon$.

Using the continuity at u or v, it is clear that $|f(u) - f(v)| < \varepsilon$.

Clearly, $\exists\delta_\varepsilon^* > 0, \forall u, v \in [a, p], |u - v| < \delta_\varepsilon^* \Longrightarrow |f(u) - f(v)| < \varepsilon$....(ii)

From (i) and (ii), it is evident that $p \in S$.

Hence $[a, y) \subseteq S$.

By induction over the continuum, we conclude that $S = [a, y)$.

Clearly, $\forall\varepsilon > 0, \exists\delta_\varepsilon > 0, \forall u, v \in [a, b), |u - v| < \delta_\varepsilon \Longrightarrow |f(u) - f(v)| < \varepsilon$.

A similar argument applies to the continuity of f at b as well.

Therefore, $\forall\varepsilon > 0, \exists\delta_\varepsilon > 0, \forall u, v \in [a, b], |u - v| < \delta_\varepsilon \Longrightarrow |f(u) - f(v)| < \varepsilon$.

Hence, the function $f: [a, b] \longrightarrow \mathbb{R}$ is uniformly continuous.

Conclusion

The conclusion of this article is that mathematical induction can be applied to a statement that holds true for all natural numbers. Furthermore, transfinite induction can be applied to a statement that holds true for all ordinal numbers. However, induction over the continuum cannot yet be applied to a specific statement, but it can be said that something is proven true if it holds for every point in the interval $[a, b)$. The suggestion that can be drawn from this article is to develop a new method of induction in mathematics in addition to the three types of induction discussed earlier.

References

Andrew. R., 2018., *Axioms and Set Theory: A first course in Set Theory*., ISBN 978-0-9938485-0-6.

Bartle. R. G., 2011., *Introduction to Real Analysis; fourth edition*., John Wiley & Sons, Inc.

Goldrei.D., 1996., *Classic Set Theory: For Guided Independent Study*., Chapman&Hall/CRC.

Herstein. I.N., 1986., *Abstract Algebra*., Macmillan Publishing Company, New York.

Hungerford. T., 1974., *Algebra: Graduate Texts in Mathematics*., Springer.

Jech. T., 2003., *Set Theory: The Third Millenium Edition revised and expanded*., Springer.

Kalantari.I., 2007., *Induction over the Continuum*., https://www.researchgate.net/publication/226970047.

Royden., 2010., *Real Analysis: fourth edition*., China Machine Press.

Singh. N.K., Yadav. A. K., 2017., *Inductive and Deductive Methods in Mathematics Teaching*., Int. Journal of Engineering Research and Application, ISSN: 2248-9622., Vol.7, Issue 11, (Part 2) November 2017, pp.19-22.

Author Information

Denik Agustito*
https://orcid.org/0009-0009-4463-9651
Universitas Sarjanawiyata Tamansiswa
Jl. Batikan UH-III/1043 Yogyakarta 55167
Indonesia
Contact e-mail: *agustito@ustjogja.ac.id*

Sukiyanto
https://orcid.org/0000-0001-5381-6087
Universitas Sarjanawiyata Tamansiswa
Jl. Batikan UH-III/1043 Yogyakarta 55167
Indonesia
Contact e-mail: *sukiyanto.math@ustjogja.ac.id*

Krida Singgih Kuncoro
https://orcid.org/0000-0001-6173-1251
Universitas Sarjanawiyata Tamansiswa
Jl. Batikan UH-III/1043 Yogyakarta 55167
Indonesia
Contact e-mail: *krida.kuncoro@ustjogja.ac.id*