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An Edge Irregular Reflexive $k-$labeling of Comb Graphs with Additional 2 Pendants

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Abstrak
Misalkan G adalah graf terhubung, sederhana, dan tak berarah dengan himpunan titik $V(G)$ dan himpunan sisi $E(G)$. Misalkan k adalah suatu bilangan asli dan untuk sebarang graf G didefinisikan pelabelan$-k$ total ρ sehingga himpunan titik pada graf G dilabeli dengan $\{0, 2, 4, \ldots, 2k_v\}$ dan himpunan sisinya dilabeli dengan $\{1, 2, 3, \ldots, k_e\}$, dengan $k = maks\{2k_v, k_e\}$. Pelabelan$-k$ total ρ disebut sebagai pelabelan$-k$ total tak reguler refleksif sisi jika setiap dua sisi pada graf G memiliki bobot sisi yang berbeda, bobot sisi adalah jumlahan dari label sisi dan dua label titik yang insiden dengan sisi tersebut. Nilai terkecil k sehingga graf G dapat dilabeli dengan pelabelan$-k$ total tak reguler refleksif sisi disebut kekuatan sisi refleksif. Pada paper ini akan dibahas mengenai kekuatan sisi refleksif pada beberapa graf sisir.
Kata kunci: Pelabelan$-k$ total tak reguler refleksif sisi, kekuatan sisi refleksif, graf sisir

Abstract
Let G be a connected, simple, and undirrected graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set. Let k be a natural number. For graph G we define a total $k-$labeling ρ such that the vertices of graph G are labeled with $\{0, 2, 4, \ldots, 2k_v\}$ and the edges of graph G are labeled with $\{1, 2, 3, \ldots, k_e\}$, where $k = max\{2k_v, k_e\}$. Total $k-$labeling ρ called an edge irregular reflexive $k-$ labeling if every two distinct edges of graph G have distinct edge weights, where the edge weight is defined as the sum of the label of that edge and the label of the vertices that are incident to this edge. The minimum k such that G has an edge irregular reflexive $k-$labeling called the reflexive edge strength of G. In this paper we determine the reflexive edge strength of some comb graphs.
Keywords: Edge irregular reflexive $k-$labeling, reflexive edge strength, comb graphs

1. INTRODUCTION

All graphs considered in this paper are finite, connected, and undirected. Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Labeling

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graphs still become an exciting research topic. In 1988, Chartrand et al. [4] defined irregular assignment as a function of the edge set of graph G to the set $\{1, 2, \ldots, k\}$ such that every two distinct vertices have different vertex weights. The vertex weight is defined as the sum of labels of edges that are incident to the vertex. Moreover, Bac'a et.al introduced edge irregular total $k-$labeling of a graph G as a function from union of edge set and vertex set of graph G to the set $\{1, 2, \ldots, k\}$ such that every two distinct edges have different edge weights. The edge weight is defined as the sum of edge label and all vertex labels that are incident to that edge.

Inspired by irregular assignment and edge irregular total $k-$labeling, Ryan et al. defined the edge irregular reflexive $k-$labeling. The total $k-$labeling ρ on graph G is a combination between vertex labeling $\rho_v : V(G) \to \{0, 2, \ldots, 2k_v\}$ and edge labeling $\rho_e : E(G) \to \{1, 2, \ldots, k_e\}$ with $k = max\{k_e, 2k_v\}$. The total $k-$labeling ρ is called an edge irregular reflexive $k-$labeling if every two different edges $u_1v_1, u_2v_2 \in E(G)$ have different edge weights, that is $wt_\rho(u_1v_1) = \rho_v(u_1) + \rho_e(u_1v_1) + \rho_v(v_1) \neq wt_\rho(u_2v_2) = \rho_v(u_2) + \rho_e(u_2v_2) + \rho_v(v_2)$. The smallest positive integer k such that the graph G has the edge irregular reflexive $k-$labeling is called the reflexive edge strength and denoted by $res(G)$. For more research for reflexive edge strength see [1], [5], [6], [8], [9], and [10]. In this paper we will study about the reflexive edge strength of comb graphs with additional 2 pendants and double comb graphs with additional 2 pendants. Moreover, additional proof is provided for the theorem of circulant graphs in [2]. Here given lemma proven by Ryan et.al. [7].

Lemma 1.1. *For every graph G, it follows that*

$$res(G) \geq \begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & |E(G)| \not\equiv 2, 3 (mod\ 6) \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & |E(G)| \equiv 2, 3 (mod\ 6). \end{cases}$$

Furthermore, Bača et.al [3] proposed the following conjecture:

conjecture 1.2. *Let G be a simple graph with maximum degree $\Delta(G)$. Then*

$$res(G) = maks\left\{ \left(\frac{\Delta(G) + 2}{2} \right), \left(\frac{|E(G)|}{3} \right) + r \right\},$$

where $r = 1$ for $|E(G)| \equiv 2, 3 (mod\ 6)$, and zero otherwise.

2. RESULT AND DISCUSSION

2.1. **The Comb Graphs with Additional 2 Pendants.** The comb graphs with additional 2 pendants, denoted by $Comb_n^{2+}$, is defined as the graph with vertex set

$$V(Comb_n^{2+}) = \{u_i \mid 1 \leq i \leq n\} \cup \{v_j^i \mid 2 \leq i \leq n-1\},$$

and edge set

$$E(Comb_n^{2+}) = \{u_iu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{u_iv_j^i \mid 2 \leq i \leq n-1\}.$$

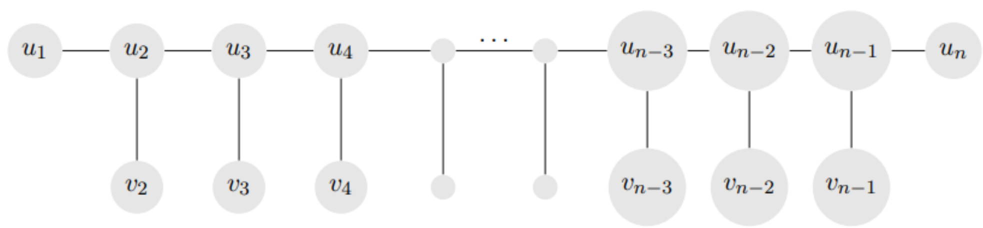

FIGURE 1. Graph $Comb_n^{2+}$

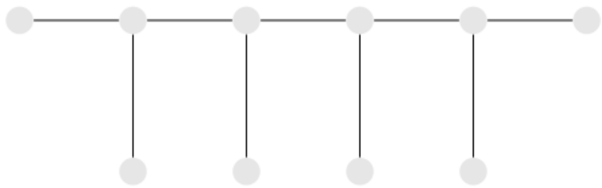

FIGURE 2. Graph $Comb_6^{2+}$

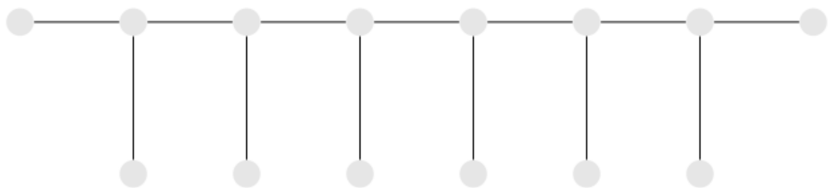

FIGURE 3. Graph $Comb_8^{2+}$

The comb graphs with additional 2 pendants with n vertices is shown in Figure 1. Moreover, Figure 2 Figure 3 are examples of comb graphs with additional 2 pendants $Comb_6^{2+}$ and $Comb_8^{2+}$.

The following theorem is about the reflexive edge strength of comb graphs with additional 2 pendants.

Theorem 2.1. *Let $Comb_n^{2+}$ be a comb graph with additional 2 pendants. If $n \geq 3$, then*

$$res(Comb_n^{2+}) = \begin{cases} \lceil \frac{2n-3}{3} \rceil, & 2n-3 \not\equiv 3(mod\ 6), \\ \lceil \frac{2n-3}{3} \rceil + 1, & 2n-3 \equiv 3(mod\ 6). \end{cases}$$

PROOF. The number of edges of comb graphs with additional 2 pendants $Comb_n^{2+}$, for $n \geq 3$ is $2n-3$. By Lemma 1.1, the lower bound for res of the comb graphs with additional 2 pendants is as follows,

$$res(Comb_n^{2+}) = k \geq \begin{cases} \lceil \frac{2n-3}{3} \rceil, & 2n-3 \not\equiv 3(\text{mod } 6), \\ \lceil \frac{2n-3}{3} \rceil + 1, & 2n-3 \equiv 3(\text{mod } 6). \end{cases}$$

To prove the upper bound we define an edge irregular reflexive $k-$labeling ρ as follows.

All vertices u_i for $1 \leq i \leq n$ are labeled with even integers in the following ways:

$$\rho_v(u_i) = 2\lfloor \frac{i}{3} \rfloor.$$

All vertices v_i for $2 \leq i \leq n-1$ are labeled with even integers in the following ways:

$$\rho_v(v_i) = 2\lfloor \frac{i-1}{3} \rfloor.$$

All edges u_iu_{i+1} for $1 \leq i \leq n-1$ are labeled as follows:

$$\rho_e(u_iu_{i+1}) = 2\lceil \frac{i}{3} \rceil - 1.$$

All edges u_iv_i for $2 \leq i \leq n-1$ are labeled as follows:

$$\rho_e(u_iv_i) = 2\lceil \frac{i-1}{3} \rceil.$$

The biggest vertex label is $\rho_v(u_n) = 2\lfloor \frac{n}{3} \rfloor$ and the biggest edge labels are $\rho_e(u_{n-1}u_n) = 2\lceil \frac{n-1}{3} \rceil - 1$ and $\rho_e(u_{n-1}v_{n-1}) = 2\lceil \frac{(n-1)-1}{3} \rceil = 2\lceil \frac{n-2}{3} \rceil$. We will prove that

$$res(Comb_n^{2+}) = k \leq \begin{cases} \lceil \frac{2n-3}{3} \rceil, & 2n-3 \not\equiv 3(\text{mod } 6), \\ \lceil \frac{2n-3}{3} \rceil + 1, & 2n-3 \equiv 3(\text{mod } 6). \end{cases}$$

Based on the theorem, we will obtain 2 cases:

(i) For $2n-3 \equiv 3(\text{mod } 6)$.
Let $n = 3x+3$, for some $x \in \mathbb{N}$. The vertex label is

$$\rho_v(u_n) = 2\lfloor \frac{n}{3} \rfloor = 2\lfloor \frac{3x+3}{3} \rfloor = 2x+2,$$

and the edge labels are:

$$\rho_e(u_{n-1}u_n) = 2\lceil \frac{n-1}{3} \rceil - 1 = 2\lceil \frac{(3x+3)-1}{3} \rceil - 1 = 2\lceil \frac{3x+2}{3} \rceil - 1 = 2x+1,$$

and

$$\rho_e(u_{n-1}v_{n-1}) = 2\lceil \frac{(n-2}{3} \rceil = 2\lceil \frac{(3x+3)-2}{3} \rceil = 2\lceil \frac{3x+1}{3} \rceil = 2x+2.$$

Futhermore,

$$\lceil \frac{2n-3}{3} \rceil + 1 = \lceil \frac{2(3x+3)-3}{3} \rceil + 1 = \lceil \frac{6x+3}{3} \rceil + 1 = 2x+2.$$

It is clear that the biggest label is $2\lceil \frac{(n-2)}{3} \rceil = \lceil \frac{2n-3}{3} \rceil + 1 = 2x+2$.

(ii) For $2n-3 \not\equiv 3(\text{ mod } 6)$, we consider the following two cases.

(a) Case 1. For $2n-3 \equiv 1(\text{mod } 6)$.
Let $n = 3x+2$, for some $x \in \mathbb{N}$. The vertex label is

$$\rho_v(u_n) = 2\lfloor \frac{n}{3} \rfloor = 2\lfloor \frac{3x+2}{3} \rfloor = 2x,$$

and the edge labels are:

$$\rho_e(u_{n-1}u_n) = 2\lceil \frac{n-1}{3} \rceil - 1 = 2\lceil \frac{(3x+2)-1}{3} \rceil - 1 = 2\lceil \frac{3x+1}{3} \rceil - 1 = 2x+1,$$

and

$$\rho_e(u_{n-1}v_{n-1}) = 2\lceil \frac{n-2}{3} \rceil = 2\lceil \frac{(3x+2)-2}{3} \rceil = 2\lceil \frac{3x}{3} \rceil = 2x.$$

Moreover,

$$\lceil \frac{2n-3}{3} \rceil = \lceil \frac{2(3x+2)-3}{3} \rceil = \lceil \frac{6x+1}{3} \rceil = 2x+1.$$

Thus, the biggest label is $2\lceil \frac{n-1}{3} \rceil - 1 = \lceil \frac{2n-3}{3} \rceil = 2x+1$.

(b) For $2n-3 \equiv 5(\text{mod } 6)$.
Let $n = 3x+4$, for some $x \in \mathbb{N}$. The vertex label is

$$\rho_v(u_n) = 2\lfloor \frac{n}{3} \rfloor = 2\lfloor \frac{3x+4}{3} \rfloor = 2x+2,$$

and the edge labels are:

$$\rho_e(u_{n-1}u_n) = 2\lceil \frac{n-1}{3} \rceil - 1 = 2\lceil \frac{(3x+4)-1}{3} \rceil - 1 = 2\lceil \frac{3x+3}{3} \rceil - 1 = 2x+2,$$

and

$$\rho_e(u_{n-1}v_{n-1}) = 2\lceil \frac{n-2}{3} \rceil = 2\lceil \frac{(3x+4)-2}{3} \rceil = 2\lceil \frac{3x+2}{3} \rceil = 2x+1.$$

Furthermore,

$$\lceil \frac{2n-3}{3} \rceil = \lceil \frac{2(3x+2)-3}{3} \rceil = \lceil \frac{6x+1}{3} \rceil = 2x+1.$$

Therefore, the biggest label is $2\lceil \frac{n-1}{3} \rceil - 1 = \lceil \frac{2n-3}{3} \rceil = 2x+1$.

Moreover, the edge weight of all edges under the total $k-$labeling ρ for $1 \le i \le n$ are as follows.
The edge weight of u_iu_{i+1} is $wt(u_iu_{i+1}) = \rho_v(u_i) + \rho_e(u_iu_{i+1}) + \rho_v(u_{i+1})$, for $1 \le i \le n$. Hence

$$wt(u_iu_{i+1}) = 2\lfloor \frac{i}{3} \rfloor + 2\lceil \frac{i}{3} \rceil - 1 + 2\lfloor \frac{(i+1)}{3} \rfloor = 2\left(\lfloor \frac{i}{3} \rfloor + \lceil \frac{i}{3} \rceil + \lfloor \frac{i+1}{3} \rfloor \right) - 1.$$

Furhermore, the edge weight of u_iv_i is $wt(u_iv_i) = \rho_v(u_i) + \rho_e(u_iv_i) + \rho_v(v_i)$ for $1 \le i \le n-1$. Therefore,

$$wt(u_iv_i) = 2\lfloor \frac{i}{3} \rfloor + 2\lceil \frac{i-1}{3} \rceil + 2\lfloor \frac{i-1}{3} \rfloor = 2\left(\lfloor \frac{i}{3} \rfloor + \lceil \frac{i-1}{3} \rceil + \lfloor \frac{i-1}{3} \rfloor \right).$$

It is clear that the edge weights of all edges in $Comb_n^{2+}$ are distinct integers. Thus, the total $k-$labeling ρ is an edge irregular reflexive $k-$labeling of $Comb_n^{2+}$ and k is the reflexive edge strength of $Comb_n^{2+}$. Figure 4 is the edge irregular reflexive $8-$labeling of a comb graph with 2 additional pendants $Comb_8^{2+}$.

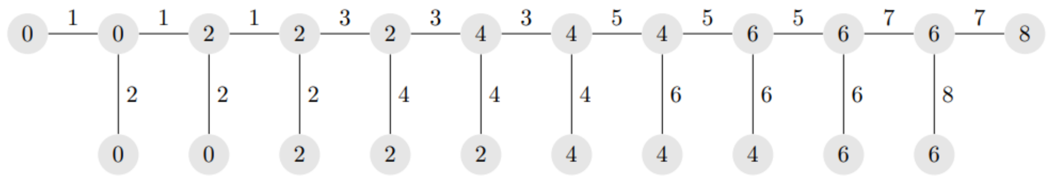

FIGURE 4. The Edge Irregular Reflexive $8-$labeling of Graph $Comb_8^{2+}$

2.2. **The Double Comb Graphs with Additional 2 Pendants.** The double comb graphs with additional 2 pendants, denoted by $DComb_n^{2+}$, is defined as the graph with vertex set

$$V(DComb_n^{2+}) = \{u_i \mid 1 \le i \le n\} \cup \{v_i \mid 2 \le i \le n-1\} \cup \{w_i \mid 2 \le i \le n-1\}$$

and edge set

$$E(DComb_n^{2+}) = \{u_iu_{i+1} \mid 1 \le i \le n-1\} \cup \{u_iv_i \mid 2 \le i \le n-1\} \cup \{u_iw_i \mid 2 \le i \le n-1\}.$$

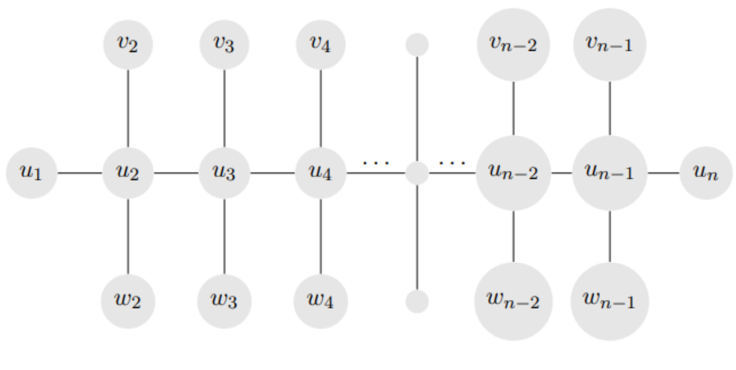

FIGURE 5. Graph $DComb_n^{2+}$

The double comb graphs with additional 2 pendants with n vertices is shown in Figure 5.

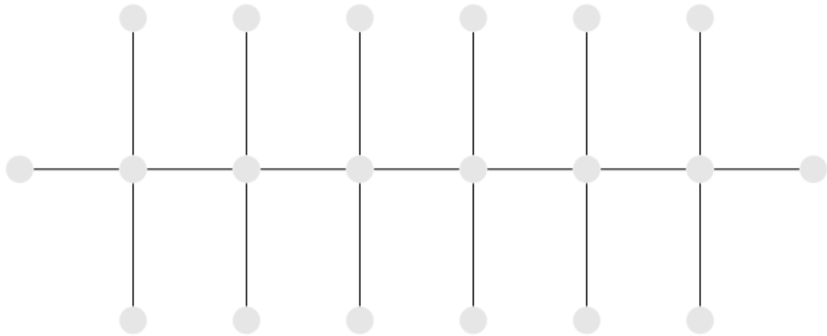

FIGURE 6. Graph $DComb_8^{2+}$

Figure 6 is an example of double comb graph with additional 2 pendants $DComb_8^{2+}$. The number of edges of double comb graph with additional 2 pendants is $3n-5$ for $n \geq 3$. Note that $3n-5 \pmod 6 = 3n+1 \pmod 6$, for $n \in \mathbb{N}$ and

$$3n \pmod 6 = \begin{cases} 0, & n \text{ even}, \\ 3, & n \text{ odd}. \end{cases}$$

It means $3n-5 \pmod 6 = 3n+1 \pmod 6 = 1$ or 4. On the other words $3n-5 \pmod 6 \not\equiv 2, 3$. Hence, we have the following theorem.

Theorem 2.2. *Let $DComb_n^{2+}$ be a double comb graph with additional 2 pendants. If $n \geq 3$, then*

$$res(DComb_n^{2+}) = \lceil \frac{3n-5}{3} \rceil.$$

PROOF. We know that the graph $DComb_n^{2+}$, for $n \geq 3$ has $3n-5$ edges. By Lemma 1.1, the lower bound for res of comb graph with additional 2 pendants $DComb_n^{2+}$ is as follows:

$$res(DComb_n^{2+}) = k \geq \lceil \frac{3n-5}{3} \rceil.$$

The next step, we determine the upper bound of $res(DComb_n^{2+})$ by defining the function ρ. All vertices u_i for $1 \leq i \leq n$ are labeled with even integers as follows:

$$\rho_v(u_i) = 2\lfloor \frac{i-1}{2} \rfloor.$$

All vertices v_i for $2 \leq i \leq n-1$ are labeled with even integers in the following ways:

$$\rho_v(v_i) = 2\lfloor \frac{i}{2} \rfloor.$$

Furthermore, all vertices u_i for $2 \leq i \leq n-1$ are labeled with even integers as follows:

$$\rho_v(w_i) = \begin{cases} 0, & i = 2, \\ 2\lfloor \frac{i-1}{2} \rfloor, & 3 \leq i \leq n-1. \end{cases}$$

Moreover, all edges u_iu_{i+1}, for $1 \leq i \leq n-1$, are labeled below:

$$\rho_e(u_iu_{i+1}) = i.$$

All edges u_iv_i, for $2 \leq i \leq n-1$, are labeled in the following ways:

$$\rho_e(u_iv_i) = 2\lfloor \frac{i}{2} \rfloor - 1.$$

All edges u_iw_i, for $2 \le i \le n-1$, are labeled as follows:

$$\rho_e(u_iw_i) = \begin{cases} 2, & i = 2, \\ 2\lceil \frac{i-2}{2} \rceil, & 3 \le i \le n-1. \end{cases}$$

The biggest vertex label is $\rho_v(v_{n-1}) = 2\lfloor \frac{n-1}{2} \rfloor$ and the biggest edge label is $\rho_e(u_{n-1}u_n) = n$. We will prove that $res(DComb_n^{2+}) = k \le \lceil \frac{3n-5}{3} \rceil$ and we consider two cases.

(i.) For n even.
Let $n = 2x$, for some $x \in \mathbb{N}$. The vertex label is

$$\rho_v(v_{n-1}) = 2\lfloor \frac{n-1}{2} \rfloor = 2\lfloor \frac{2x-1}{2} \rfloor = 2\lfloor \frac{2x}{2} \rfloor + \lfloor \frac{-1}{2} \rfloor = 2(x+0) = 2x,$$

and the edge label is

$$\rho_e(u_iu_{i+1}) = n-1 = 2x-1.$$

Moreover, since $3n-5 \pmod 6 \; = 3n+1 \pmod 6$, we have

$$\lceil \frac{3n-5}{3} \rceil = \lceil \frac{3n+1}{3} \rceil = \lceil \frac{3(2x)+1}{3} \rceil = \lceil \frac{6x+1}{3} \rceil = 2x+1.$$

Thus, we obtained that the vertex label and the edge label always less than $\lceil \frac{3n-5}{3} \rceil$.

(ii.) For n odd.
Let $n = 2x+1$, for some $x \in \mathbb{N}$. The vertex label is

$$\rho_v(v_{n-1}) = 2\lfloor \frac{n-1}{2} \rfloor = 2\lfloor \frac{(2x+1)-1}{2} \rfloor = 2\lfloor \frac{2x}{2} \rfloor = 2x,$$

and the edge label is

$$\rho_e(u_iu_{i+1}) = n-1 = (2x+1)-1 = 2x.$$

Moreover, since $3n-5 \pmod 6 \; = 3n+1 \pmod 6$, we have

$$\lceil \frac{3n-5}{3} \rceil = \lceil \frac{3n+1}{3} \rceil = \lceil \frac{3(2x+1)+1}{3} \rceil = \lceil \frac{6x+2}{3} \rceil = 2x+1.$$

Thus, we obtained that the vertex label and the edge label always less than $\lceil \frac{3n-5}{3} \rceil$.

Therefore, the edge weights of graph $DComb_n^{2+}$ under the labeling ρ are the following:

The edge weight of u_iu_{i+1} is $wt(u_iu_{i+1}) = \rho_v(u_i) + \rho_e(u_iu_{i+1}) + \rho_v(u_{i+1})$, for $1 \le i \le n-1$. Therefore,

$$wt(u_iu_{i+1}) = 2\lfloor \frac{i-1}{2} \rfloor + i + 2\lfloor \frac{(i-1)+1}{2} \rfloor = 2\left(\lfloor \frac{i-1}{2} \rfloor + \lfloor \frac{i}{2} \rfloor \right) + i.$$

Moreover, the edge weight of u_iv_i is $wt(u_iv_i) = \rho_v(u_i) + \rho_e(u_iv_i) + \rho_v(v_i)$ for $1 \le i \le n-1$. Hence,

$$wt(u_iv_i) = 2\lfloor \frac{i-1}{2} \rfloor + 2\lfloor \frac{i}{2} \rfloor - 1 + 2\lfloor \frac{i}{2} \rfloor = 2\left(\lfloor \frac{i-1}{2} \rfloor + 2\lfloor \frac{i}{2} \rfloor \right) - 1.$$

The edge weight of u_iw_i is $wt(u_iw_i) = \rho_v(u_i) + \rho_e(u_iw_i) + \rho_v(w_i)$. Thus, for $i = 2$, we have

$$wt(u_iw_i) = 2\lfloor \frac{i-1}{2} \rfloor + 2 + 0 = 2\lfloor \frac{2-1}{2} \rfloor + 2 = 2\lfloor \frac{1}{2} \rfloor + 2 = 2,$$

and

$$wt(u_iw_i) = 2\lfloor \frac{i-1}{2} \rfloor + 2\lceil \frac{i-2}{2} \rceil + 2\lfloor \frac{i}{2} \rfloor = 2\left(\lfloor \frac{i}{2} \rfloor + 2\lfloor \frac{i-1}{2} \rfloor + 2\lceil \frac{i-2}{2} \rceil \right),$$

for $3 \le i \le n-1$.

It is already showed that the edge weights of all edges in $DComb_n^{2+}$ are distinct integers. Thus, the total $k-$labeling ρ is an edge irregular reflexive $k-$labeling of $DComb_n^{2+}$ and k is the reflexive edge strength of $DComb_n^{2+}$.

Figure 7 is the edge irregular reflexive 10−labeling of a double comb graph with 2 additional pendants $DComb_{11}^{2+}$.

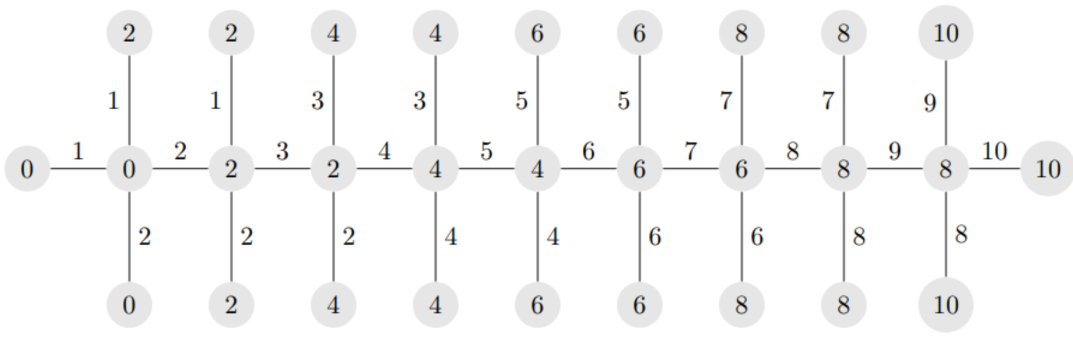

FIGURE 7. The Edge Irregular Reflexive 10−labeling of Graph $DComb_{11}^{2+}$

3. CIRCULANT GRAPHS

Let m, n are possitive integers and S be the set $S = \{s_1, s_2, \dots, s_m\}$, with $1 \leq s_1 \leq s_2 \leq \dots \leq s_m \leq \frac{n}{2}$. By circulant graph $C_n(S)$ we mean a graph with n vertices with vertex set

$$V(C_n(s_1, s_2, \dots, s_m) = \{x_1, x_2, \dots, x_{n-1}\}$$

and edge set

$$E(C_n(s_1, s_2, \dots, s_m) = \{x_i x_{i+s_j} | i = 1, 2, \dots, n, j = 1, 2, \dots, m \text{ and } i + s_j \bmod n\}.$$

Here are example of some circulant graphs. Figure 1 and Figure 2 are example of circulant graph $C_6(2, 3)$ and $C_8(1, 3, 4)$ with vertex set

$$V(C_6(2, 3)) = \{x_1, x_2, x_3, x_4, x_5, x_6\},$$

$$V(C_8(1, 3, 6)) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\},$$

and edge set

$$E(C_6(2, 3)) = \{x_1x_3, x_2x_4, x_3x_5, x_4x_6, x_5x_1, x_6x_2, x_1x_4, x_2x_5, x_3x_6, \},$$

$$E(C_8(1, 3, 6)) = \{x_1x_4, x_2x_5, x_3x_6, x_4x_7, x_5x_8, x_6x_1, x_7x_2, x_8x_3, x_1x_7, x_2x_8, x_3x_1, x_4x_2, x_5x_3, x_6x_4, x_7x_5, x_8x_6\}.$$

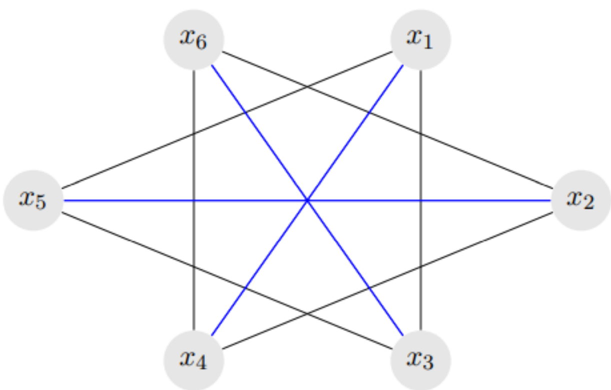

FIGURE 8. Circulant Graph $C_6(2, 3)$

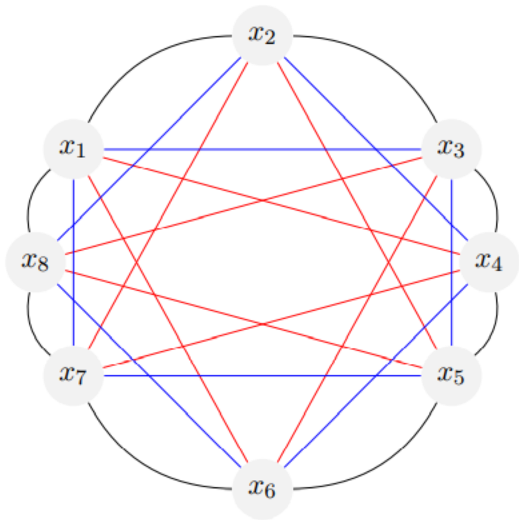

FIGURE 9. Circulant Graph $C_8(1,3,6)$

In this section, we will discuss about reflexive edge strength of circulant graphs. In [2] it is given the following theorem without proof. Here we give the complete proof.

Theorem 3.1. *Let $C_n(1,2)$ be a circulant graph. For $4 \leq n \leq 6$ it follows that the*

$$res(C_n(1,2)) = \begin{cases} 4, & n = 4, \\ 5, & n = 5, 6, \end{cases}$$

PROOF The number of edges of graphs $C_n(1,2)$, for $n \geq 5$ is $2n$. Suppose reflexive edge strength of circulant graph $C_4(1,2)$ is 3. Since $res(C_4(1,2)) = 3$, the circulant graph $C_4(1,2)$ can be labeled with irregular reflexive 3-labeling and all the edges have different edge weights. The number of edges of graph $C_4(1,2)$ are 6 and all vertices must be labeled with even numbers.

1. If all vertices of graph $C_4(1,2)$ are only labeled with 0 or 2, then the edges of graph $C_4(1,2)$ can be labeled with at least $1, 2, \ldots, 6$.
2. If one vertex is labeled with 0 and 3 others are labeled with 2, then there are 3 edges whose endpoints are labeled with 0 and 2. Therefore, all three edges must be labeled with $1, 2, 3$. Moreover, there are 3 edges whose endpoints are only labeled with 2. Hence, all three edges also must be labeled with $1, 2, 3$. But, there will be 2 edges that have edge weights 5, that is $0 + 3 + 2$ and $2 + 1 + 2$.
3. If one vertex is labeled with 2 and 3 others are labeled with 0, then there are 3 edges whose endpoints are labeled with 0 and 2. Hence, all three edges must be labeled with $1, 2, 3$. Furthermore, there are 3 edges whose endpoints are only labeled with 0. Therefore, all three edges also must be labeled with $1, 2, 3$. But, there will be 2 edges that have edge weights 3, that is $0 + 1 + 2$ and $0 + 3 + 0$.
4. If two vertices are labeled with 2 and 2 others are labeled with 0, then there are 4 edges whose endpoints are labeled with 0 and 2. Consequently, those edges must be labeled with $1, 2, 3, 4$.

Hence, we can conclude that $res(C_4(1,2)) \neq 3$. Therefore, the reflexive edge strength of graph $C_4(1,2)$ is 4.

Suppose the reflexive edge strength of circulant graph $C_5(1,2)$ is 3. Since $res(C_4(1,2)) = 3$, the circulant graph $C_4(1,2)$ can be labeled with irregular reflexive 3-labeling and all the edges have different edge weights. The number of edges of graph $C_5(1,2)$ are 10 and all vertices must be labeled with even numbers.

1. If all vertices of graph $C_5(1,2)$ are only labeled with 0 or 2, then the edges of graph $C_4(1,2)$ can be labeled with at least $1,2,\ldots,10$.
2. If 4 vertices are labeled with a and one vertex is labeled with 0, for $a \neq b$ and $a,b \in \{0,2,4\}$, then there are 6 edges that are incident to two vertices labeled with a. Hence, those edges must be labeled with $1,2,\ldots,6$.
3. If two vertices are labeled with a and 3 others are labeled with b, where a and b are distinct elements for $a,b \in \{0,2,4\}$, then there are 6 edges that are incident to vertex labeled with a and b. Therefore, that the six edges must be labeled with $1,2,\ldots,6$.
4. If one vertex is labeled with a, one vertex is labeled with b, and 3 others are labeled with c, where a,b and c are distinct elements for $a,b,c \in \{0,2,4\}$, then there are 3 edges that are incident to vertex labeled with a and $c, 3$ edges that are incident to vertex labeled with b and $c, 3$ edges that are incident to two vertices labeled with c. All those edges can be labeled with $1,2,3,4$. Hence, the edge weights are obtained as follows:
   i. For $a=0, b=2$ and $c=4$.
   The edge weights that are incident to vertex labeled with a and c are in $\{5,6,7,8\}$.
   The edge weights that are incident to vertex labeled with b and c are in $\{7,8,9,10\}$.
   The edge weights that are incident to two vertices labeled with c are in $\{9,10,11,12\}$.

   ii. For $a=0, b=4$ and $c=2$.
   The edge weights that are incident to vertex labeled with a and c are in $\{3,4,5,6\}$.
   The edge weights that are incident to vertex labeled with b and c are in $\{7,8,9,10\}$.
   The edge weights that are incident to two vertices labeled with c are in $\{5,6,7,8\}$.

   iii. For $a=2, b=0$ and $c=4$.
   The edge weights that are incident to vertex labeled with a and c are in $\{7,8,9,10\}$.
   The edge weights that are incident to vertex labeled with b and c are in $\{5,6,7,8\}$.
   The edge weights that are incident to two vertices labeled with c are in $\{9,10,11,12\}$.

   iv. For $a=2, b=4$ and $c=0$.
   The edge weights that are incident to vertex labeled with a and c are in $\{3,4,5,6\}$.
   The edge weights that are incident to vertex labeled with b and c are in $\{5,6,7,8\}$.
   The edge weights that are incident to two vertices labeled with c are in $\{1,2,3,4\}$.

   v. For $a=4, b=0$ and $c=2$.
   The edge weights that are incident to vertex labeled with a and c are in $\{7,8,9,10\}$.
   The edge weights that are incident to vertex labeled with b and c are in $\{3,4,5,6\}$.
   The edge weights that are incident to two vertices labeled with c are in $\{5,6,7,8\}$.

   vi. For $a=4, b=2$ and $c=0$.
   The edge weights that are incident to vertex labeled with a and c are in $\{5,6,7,8\}$.
   The edge weights that are incident to vertex labeled with b and c are in $\{3,4,5,6\}$.
   The edge weights that are incident to two vertices labeled with c are in $\{1,2,3,4\}$.

   Consequently, there are always edges that have the same edge weights for any a,b and c.
5. If one vertex is labeled with $a, 2$ vertices are labeled with $b, 2$ vertices are labeled with c where a,b,c are distinct elements and $a,b,c \in \{0,2,4\}$, then there are 2 edges that are incident to vertex labeled with a and $c, 2$ edges that are incident to vertex labeled with a and $b, 4$ edges that are incident to vertex labeled with b and $c, 1$ edge that is incident to two vertices labeled with $b, 1$ edge that is incident to two vertices labeled with c. All those edges can be labeled with $1,2,3,4$. Thus, the edge weights are obtained as follows:

(i.) For $a = 0, b = 2$ and $c = 4$.
Two edges that are incident to vertex labeled with a and b have edge weights in $\{3, 4, 5, 6\}$.
Two edges that are incident to vertex labeled with a and c have edge weights in $\{5, 6, 7, 8\}$.
Four edges that are incident to vertex labeled with b and c have edge weights in $\{7, 8, 9, 10\}$.
One edge that is incident to two vertices labeled with b has edge weights in $\{5, 6, 7, 8\}$.
One edge that is incident to two vertices labeled with c has edge weights in $\{9, 10, 11, 12\}$.

(ii.) For $a = 0, b = 4$ and $c = 2$.
Two edges that are incident to vertex labeled with a and b have edge weights in $\{5, 6, 7, 8\}$.
Two edges that are incident to vertex labeled with a and c have edge weights in $\{3, 4, 5, 6\}$.
Four edges that are incident to vertex labeled with b and c have edge weights in $\{7, 8, 9, 10\}$.
One edge that is incident to vertex labeled with b and b has edge weights in $\{9, 10, 11, 12\}$.
One edge that is incident to two vertices labeled with c has edge weights in $\{5, 6, 7, 8\}$.

(iii.) For $a = 2, b = 0$ and $c = 4$.
Two edges that are incident to vertex labeled with a and b have edge weights in $\{3, 4, 5, 6\}$.
Two edges that are incident to vertex labeled with a and c have edge weights in $\{7, 8, 9, 10\}$.
Four edges that are incident to vertex labeled with b and c have edge weights in $\{5, 6, 7, 8\}$.
One edge that is incident to two vertices labeled with b has edge weights in $\{1, 2, 3, 4\}$.
One edge that is incident to two vertices labeled with c has edge weights in $\{9, 10, 11, 12\}$.

(iv.) For $a = 4, b = 0$ and $c = 2$.
Two edges that are incident to vertex labeled with a and b have edge weights in $\{7, 8, 910\}$.
Two edges that are incident to vertex labeled with a and c have edge weights in $\{3, 4, 5, 6\}$.
Four edges that are incident to vertex labeled with b and c have edge weights in $\{5, 6, 7, 8\}$.
One edge that is incident to two vertices labeled with b has edge weights in $\{9, 10, 11, 12\}$.
One edge that is incident to two vertices labeled with c has edge weights in $\{1, 2, 3, 4\}$.

(v.) For $a = 4, b = 0$ and $c = 2$.
Two edges that are incident to vertex labeled with a and b have edge weights in $\{5, 6, 7, 8\}$.
Two edges that are incident to vertex labeled with a and c have edge weights in $\{7, 8, 9, 10\}$.

Four edges that are incident to vertex labeled with b and c have edge weights in $\{3,4,5,6\}$.
One edge that is incident to two vertices labeled with b has edge weights in $\{1,2,3,4\}$.
One edge that is incident to two vertices labeled with c has edge weights in $\{5,6,7,8\}$.

(vi.) For $a=4, b=2$ and $c=0$.
Two edges that are incident to vertex labeled with a and b have edge weights in $\{7,8,9,10\}$.
Two edges that are incident to vertex labeled with a and c have edge weights in $\{5,6,7,8\}$.
Four edges that are incident to vertex labeled with b and c have edge weights in $\{3,4,5,6\}$.
One edge that is incident to vertex labeled with b and b has edge weights in $\{5,6,7,8\}$.
One edge that is incident to vertex labeled with c and c has edge weights in $\{1,2,3,4\}$.

Consequently, for any a, b, and c, it will always be obtained the same edge weights.

We get the reflexive edge strength of circulant graphs $C_5(1,2) \neq 4$. Therefore, $res(C_5(1,2)) = 5$.

Next, we will show that the reflexive edge strength of graphs $C_6(1,2)$ is 5. Suppose $res(C_5(1,2)) = 4$. It means that the graph $C_6(1,2)$ can be labeled with edge irregular total $4-$labeling. Therefore, all the edges of graph $C_6(1,2)$ have different edges weights and all the vertices are labeled with even numbers. The number of edges of graph $C_6(1,2)$ are 12.

1. If all vertices of graph $C_6(1,2)$ are only labeled with 0 or 2, then the edges of graph $C_6(1,2)$ can be labeled with $1,2,\ldots,12$.
2. If one vertex is labeled with a, 5 vertices are labeled with b for $a \neq b$ and $a, b \in \{0,2,4\}$, then there are 8 edges that are incident to two vertices labeled with b. Consequently, those edges must be labeled with $1,2,\ldots,8$.
3. If 2 vertices are labeled with a and 4 others are labeled with b for $a \neq b$ and $a, b \in \{0,2,4\}$, then there are 2 possibilities.
   (i.) There are 5 edges that are incident to vertices labeled with a and b. Thus, those edges must be labeled with $1,2,\ldots,5$.
   (ii.) There are 8 edges that are incident to vertices labeled with a and b. Therefore, those edges must be labeled with $1,2,\ldots,8$.
4. If 3 vertices are labeled with a and 3 others are labeled with b, for $a \neq b$ and $a, b \in \{0,2,4\}$, then there are 2 possibilities.
   (i.) There are 6 edges that are incident to vertices labeled with a and b. Hence, those edges must be labeled with $1,2,\ldots,6$.
   (ii.) There are 8 edges that are incident to vertices labeled with a and b. Thus, those edges must be labeled with $1,2,\ldots,8$.
5. If one vertex is labeled with a, 1 vertex is labeled with b, 4 vertices are labeled with c where a, b, c are distinct elements and $a, b, c \in \{0,2,4\}$, then there are two conditions.
   (i.) There are 5 edges that are incident to two vertices labeled with c. Therefore, those edges must be labeled with $1,2,\ldots,5$.
   (ii.) There are 4 edges that are incident to vertices labeled with a and c, 4 edges that are incident to vertices labeled with b and c, 4 edges that are incident to two vertices labeled with c.
      (a.) For $a=0, b=2$ and $c=4$.
      The edge weights of four edges that have end vertices labeled with a and c

are in $\{5, 6, 7, 8\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{7, 8, 9, 10\}$.
The edge weights of three edges that have end vertices labeled with c are in $\{9, 10, 11, 12\}$.

(b.) For $a = 0, b = 4$ and $c = 2$.
The edge weights of four edges that have end vertices labeled with a and c are in $\{3, 4, 5, 6\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{7, 8, 9, 10\}$.
The edge weights of three edges that have end vertices labeled with c are in $\{5, 6, 7, 8\}$.

(c.) For $a = 2, b = 0$ and $c = 4$.
The edge weights of four edges that have end vertices labeled with a and c are in $\{7, 8, 9, 10\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{5, 6, 7, 8\}$.
The edge weights of three edges that have end vertices labeled with c are in $\{9, 10, 11, 12\}$.

(d.) For $a = 2, b = 4$ and $c = 0$.
The edge weights of four edges that have end vertices labeled with a and c are in $\{3, 4, 5, 6\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{5, 6, 7, 8\}$.
The edge weights of three edges that have end vertices labeled with c are in $\{1, 2, 3, 4\}$.

(e.) For $a = 4, b = 0$ and $c = 2$.
The edge weights of four edges that have end vertices labeled with a and c are in $\{57, 8, 9, 10\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{3, 4, 5, 6\}$.
The edge weights of three edges that have end vertices labeled with c are in $\{5, 6, 7, 8\}$.

(f.) For $a = 4, b = 2$ and $c = 0$.
The edge weights of four edges that have end vertices labeled with a and c are in $\{5, 6, 7, 8\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{3, 4, 5, 6\}$.
The edge weights of three edges that have end vertices labeled with c are in $\{1, 2, 3, 4\}$.

Consequently, for any a, b, and c, it will always obtained the same edge weights.

6. If one vertex is labeled with a, 2 vertices are labeled with b, 3 vertices are labeled with c where a, b, c are distinct elements for $a, b, c \in \{0, 2, 4\}$, then there are 3 possibilities.
   (i.) There are 6 edges that have end vertices labeled with b and c. Hence, those edges at least can be labeled with $1, 2, \dots, 6$.
   (ii.) There are 5 edges that have end vertices labeled with b and c. Thus, those edges at least can be labeled with $1, 2, \dots, 5$.

(iii.) There are 2 edges that are incident to vertex labeled with a and $b, 2$ edges that are incident to vertex labeled with a and $c, 1$ edge that is incident to two vertices labeled with $b, 4$ edges that are incident to vertices labeled with b and $c, 3$ edges that are incident to two vertices labeled with c. Those edges can be labeled with $1, 2, 3, 4$. Therefore, the edge weights are obtained as follows:

(a.) For $a = 0, b = 2$ and $c = 4$.
The edge weights of two edges that have end vertices labeled with a and b are in $\{3, 4, 5, 6\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{5, 6, 7, 8\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{5, 6, 7, 8\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{7, 8, 9, 10\}$.
The edge weights of three edges that have end vertices labeled with c are in $\{9, 10, 11, 12\}$.

(b.) For $a = 0, b = 4$ and $c = 2$.
The edge weights of two edges that have end vertices labeled with a and b are in $\{5, 6, 7, 8\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{3, 4, 5, 6\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{9, 10, 11, 12\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{7, 8, 9, 10\}$.
The edge weights of three edges that have end vertices labeled with c are in $\{5, 6, 7, 8\}$.

(c.) For $a = 2, b = 0$ and $c = 4$.
The edge weights of two edges that have end vertices labeled with a and b are in $\{3, 4, 5, 6\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{7, 8, 9, 10\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{1, 2, 3, 4\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{5, 6, 7, 8\}$.
The edge weights of three edges that have end vertices labeled with c are in $\{9, 10, 11, 12\}$.

(d.) For $a = 2, b = 4$ and $c = 0$.
The edge weights of two edges that have end vertices labeled with a and b are in $\{7, 8, 9, 10\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{3, 4, 5, 6\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{9, 10, 11, 12\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{5, 6, 7, 8\}$.
The edge weights of three edges that have end vertices labeled with c are in $\{1, 2, 3, 4\}$.

(e.) For $a = 4, b = 0$ and $c = 2$.
The edge weights of two edges that have end vertices labeled with a and b are in $\{5, 6, 7, 8\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{7, 8, 9, 10\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{1, 2, 3, 4\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{3, 4, 5, 6\}$.
The edge weights of three edges that have end vertices labeled with c are in$\{5, 6, 7, 8\}$.

(f.) For $a = 4, b = 2$ and $c = 0$.
The edge weights of two edges that have end vertices labeled with a and b are in $\{7, 8, 9, 10\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{5, 6, 7, 8\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{9, 10, 11, 12\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{3, 4, 5, 6\}$.
The edge weights of three edges that have end vertices labeled with c are in $\{1, 2, 3, 4\}$.

Consequently, for any a, b, and c, it will always be obtained the same edge weights.

7. If 2 vertices are labeled with a, 2 vertices are labeled with b, and 2 others are labeled with c, where a, b, c are distinct elements for $a, b, c \in \{0, 2, 4\}$, then there are 3 possibilities.
   (i.) There are 4 edges that have end vertices labeled with a and b, 4 edges that have end vertices labeled with a and c, 4 edges that have end vertices labeled with b and c. All those edges must be labeled with $1, 2, 3, 4$. Therefore, we have edge weights as follows:
   (a.) For $a = 0, b = 2$, and $c = 4$.
   The weights of the edges that have end vertices labeled with a and b are in $\{3, 4, 5, 6\}$.
   The weights of the edges that have end vertices labeled with a and c are in $\{5, 6, 7, 8\}$.
   The weights of the edges that have end vertices labeled with b and c are in $\{7, 8, 9, 10\}$.

   (b.) For $a = 0, b = 4$, and $c = 2$.
   The weights of the edges that have end vertices labeled with a and b are in $\{5, 6, 7, 8\}$.
   The weights of the edges that have end vertices labeled with a and c are in $\{3, 4, 5, 6\}$.
   The weights of the edges that have end vertices labeled with b and c are in $\{7, 8, 9, 10\}$.

   (c.) For $a = 2, b = 0$, and $c = 4$.
   The weights of the edges that have end vertices labeled with a and b are in $\{3, 4, 5, 6\}$.
   The weights of the edges that have end vertices labeled with a and c are in $\{7, 8, 9, 10\}$.
   The weights of the edges that have end vertices labeled with b and c are in

$\{5, 6, 7, 8\}$.

(d.) For $a = 2, b = 4$, and $c = 0$.
The weights of the edges that have end vertices labeled with a and b are in $\{7, 8, 9, 10\}$.
The weights of the edges that have end vertices labeled with a and c are in $\{3, 4, 5, 6\}$.
The weights of the edges that have end vertices labeled with b and c are in $\{5, 6, 7, 8\}$.

(e.) For $a = 4, b = 0$, and $c = 2$.
The weights of the edges that have end vertices labeled with a and b are in $\{5, 6, 7, 8\}$.
The weights of the edges that have end vertices labeled with a and c are in $\{7, 8, 9, 10\}$.
The weights of the edges that have end vertices labeled with b and c are in $\{3, 4, 5, 6\}$.

(f.) For $a = 4, b = 2$, and $c = 0$.
The weights of the edges that have end vertices labeled with a and b are in $\{7, 8, 9, 10\}$.
The weights of the edges that have end vertices labeled with a and c are in $\{5, 6, 7, 8\}$.
The weights of the edges that have end vertices labeled with b and c are in $\{3, 4, 5, 6\}$.

Therefore, for any a, b and c the same weights can be hold.

(ii.) There are 1 edge that has end vertices labeled with a, 4 edges that have end vertices labeled with a and b, 2 edges that have end vertices labeled with a and c, 4 edges that have end vertices labeled with b and c, 1 edge that has end vertices labeled with c. All those edges must be labeled with $1, 2, 3, 4$. Therefore, we have edge weights as follows:

(a.) For $a = 0, b = 2$, and $c = 4$.
The edge weights of one edge that has end vertices labeled with a are in $\{1, 2, 3, 4\}$.
The edge weights of four edges that have end vertices labeled with a and b are in $\{3, 4, 5, 6\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{5, 6, 7, 8\}$.
The edge weights of four edges that have end vertices labeled with b and c in are $\{7, 8, 9, 10\}$.
The edge weights of one edge that has end vertices labeled with c are in $\{9, 10, 11, 2\}$.

(b.) For $a = 0, b = 4$, and $c = 2$.
The edge weights of one edge that has end vertices labeled with a are in $\{1, 2, 3, 4\}$.
The edge weights of four edges that have end vertices labeled with a and b are in $\{5, 6, 7, 8\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{3, 4, 5, 6\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{7, 8, 9, 10\}$.

The edge weights of one edge that has end vertices labeled with c are in $\{5, 6, 7, 8\}$.

(c.) For $a = 2, b = 0$, and $c = 4$.
The edge weights of one edge that has end vertices labeled with a are in $\{5, 6, 7, 8\}$.
The edge weights of four edges that have end vertices labeled with a and b are in $\{3, 4, 5, 6\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{7, 8, 9, 10\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{5, 6, 7, 8\}$.
The edge weights of one edge that has end vertices labeled with c are in $\{9, 10, 11, 12\}$.

(d.) For $a = 2, b = 4$, and $c = 0$.
The edge weights of one edge that has end vertices labeled with a are in $\{5, 6, 7, 8\}$.
The edge weights of four edges that have end vertices labeled with a and b are in $\{7, 8, 9, 10\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{3, 4, 5, 6\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{5, 6, 7, 8\}$.
The edge weights of one edge that has end vertices labeled with c are in $\{1, 2, 3, 4\}$.

(e.) For $a = 4, b = 0$, and $c = 0$.
The edge weights of one edge that has end vertices labeled with a are in $\{9, 10, 11, 12\}$.
The edge weights of four edges that have end vertices labeled with a and b are in $\{5, 6, 7, 8\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{7, 8, 9, 10\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{3, 4, 5, 6\}$.
The edge weights of one edge that has end vertices labeled with c are in $\{5, 6, 7, 8\}$.

(f.) For $a = 4, b = 2$, and $c = 0$.
The edge weights of one edge that has end vertices labeled with a are in $\{9, 10, 11, 12\}$.
The edge weights of four edges that have end vertices labeled with a and b are in$\{7, 8, 9, 10\}$.
The edge weights of two edges that have end vertices labeled with a and c are in $\{5, 6, 7, 8\}$.
The edge weights of four edges that have end vertices labeled with b and c are in $\{3, 4, 5, 6\}$.
The edge weights of one edge that has end vertices labeled with c are in $\{1, 2, 3, 4\}$.

Thus, the same edge weights of the edge are always obtained for any a, b and c.

(iii.) There is 1 edge that has end vertices labeled with $a, 3$ edges that have end vertices labeled with a and $b, 3$ edges that have end vertices labeled with a and $c, 1$ edge that has end vertices labeled with $b, 3$ edges that have end vertices labeled with b and $c, 1$ edge that has end vertices labeled with c. All those edges must be labeled with $1, 2, 3, 4$. Thus, we have edge weights as follows:

(a.) For $a = 0, b = 2$ and $c = 4$.
The edge weights of one edge that has end vertices labeled with a are in $\{1, 2, 3, 4\}$.
The edge weights of three edges that have end vertices labeled with a and b are in $\{3, 4, 5, 6\}$.
The edge weights of three edges that have end vertices labeled with a and c are in $\{5, 6, 7, 8\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{5, 6, 7, 8\}$.
The edge weights of three edges that have end vertices labeled with b and c are in $\{7, 8, 9, 10\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{9, 10, 11, 12\}$.

(b.) For $a = 0, b = 4$ and $c = 2$.
The edge weights of one edge that has end vertices labeled with a are in $\{1, 2, 3, 4\}$.
The edge weights of three edges that have end vertices labeled with a and b are in $\{5, 6, 7, 8\}$.
The edge weights of three edges that have end vertices labeled with a and c are in $\{3, 4, 5, 6\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{9, 10, 11, 12\}$.
The edge weights of three edges that have end vertices labeled with b and c are in $\{7, 8, 9, 10\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{5, 6, 7, 8\}$.

(c.) For $a = 2, b = 0$ and $c = 4$.
The edge weights of one edge that has end vertices labeled with a are in $\{5, 6, 7, 8\}$.
The edge weights of three edges that have end vertices labeled with a and b are in $\{3, 4, 5, 6\}$.
The edge weights of three edges that have end vertices labeled with a and c are in $\{7, 8, 9, 10\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{1, 2, 3, 4\}$.
The edge weights of three edges that have end vertices labeled with b and c are in $\{5, 6, 7, 8\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{9, 10, 11, 12\}$.

(d.) For $a = 0, b = 2$ and $c = 4$.
The edge weights of one edge that has end vertices labeled with a are in $\{5, 6, 7, 8\}$.
The edge weights of three edges that have end vertices labeled with a and b are in $\{7, 8, 9, 10\}$.
The edge weights of three edges that have end vertices labeled with a and c

are in $\{3, 4, 5, 6\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{9, 10, 11, 12\}$.
The edge weights of three edges that have end vertices labeled with b and c are in $\{5, 6, 7, 8\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{1, 2, 3, 4\}$.

(e.) For $4 = 0, b = 0$ and $c = 2$.
The edge weights of one edge that has end vertices labeled with a are in $\{9, 10, 11, 12\}$.
The edge weights of three edges that have end vertices labeled with a and b are in $\{5, 6, 7, 8\}$.
The edge weights of three edges that have end vertices labeled with a and c are in $\{7, 8, 9, 10\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{1, 2, 3, 4\}$.
The edge weights of three edges that have end vertices labeled with b and c are in $\{3, 4, 5, 6\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{5, 6, 7, 8\}$.

(f.) For $a = 4, b = 2$ and $c = 0$.
The edge weights of one edge that has end vertices labeled with a are in $\{9, 10, 11, 12\}$.
The edge weights of three edges that have end vertices labeled with a and b are in $\{7, 8, 9, 10\}$.
The edge weights of three edges that have end vertices labeled with a and c are in $\{5, 6, 7, 8\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{5, 6, 7, 8\}$.
The edge weights of three edges that have end vertices labeled with b and c are in $\{3, 4, 5, 6\}$.
The edge weights of one edge that has end vertices labeled with b are in $\{1, 2, 3, 4\}$.

Thus, the same edge weights will always be acquired for any a, b and c.

Figure 10 is the edge irregular reflexive $10-$labeling of a double comb graph with 2 additional pendants $Comb^{2+}_{11_2}$.

The following theorem has been proof in [2].

Theorem 3.2. *Let $C_n(1, 2)$ be a circulant graph. If $n \geq 10$, then*

$$res(C_n(1,2)) = \begin{cases} \lceil \frac{2n}{3} \rceil, & n \equiv 0, 2(mod\ 3) \ \textit{and}\ n \geq 10, \\ \lceil \frac{2n}{3} \rceil + 1, & n \equiv 1(mod\ 3) \ \textit{and}\ n \geq 11. \end{cases}$$

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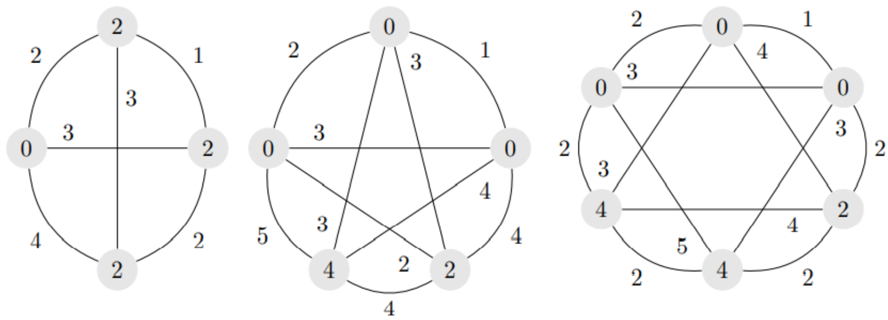

FIGURE 10. The Edge Irregular Reflexive $k-$labeling of a Circulant Graph $C_4(1,2), C_5(1,2), C_6(1,2)$

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