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Solving of Linear Volterra-Fredholm Integral Equations via Modification of Block Pulse Functions

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Abstract

A computational method based on modification of block pulse functions is proposed for solving numerically the linear Volterra-Fredholm integral equations. We obtain integration operational matrix of modification of block pulse functions on interval $[0, T)$. A modification of block pulse functions and their integration operational matrix can be reduced to a linear upper triangular system. Then, the problem under study is transformed to a system of linear algebraic equations which can be used to obtain an approximate solution of linear Volterra-Fredholm integral equations. Furthermore, the rate of convergence is $\mathcal{O}(h)$ and error analysis of the proposed method are investigated. The results show that the approximate solutions have a good of efficiency and accuracy.

Keywords: Integration Operational Matrix, Linear Volterra-Fredholm Integral Equations, εMBPFs.

1. Introduction

An integral equation is defined as an equation in which the unknown function $X(t)$ to be determined appear under one or more integral signs. The subject of integral equations is one of the most useful mathematical tools in pure and applied mathematics. It arise naturally in physics, chemistry, biology, and engineering applications modelled by initial value problems for a finite interval $[a, b]$. It also arise as representation formulas for the solutions of differential equations. Indeed, a differential equation can be replaced by an integral equation that incorporates its boundary conditions [4]. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations [8].

The Volterra-Fredholm integral equation, which is a combination of disjoint Volterra and Fredholm integrals, appears in one integral equation. The Volterra-Fredholm integral equations arise from parabolic boundary value problems, mathematical modelling of the spatio-temporal development of an epidemic, various physical, biological, and chemical applications [10, 11]. There are several techniques for approximating the solution such as moving least square method and Chebyshev polynomials [1], collocation and Galerkin methods [2], parameterized pseudospectral integration matrices [9], triangular functions [5], Taylor polynomial [12],

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and Legendre collocation method [7]. In this paper, we apply a numerical method based on the modification of block pulse functions and integration operational matrix to consider the following linear Volterra-Fredholm integral equation:

$$X(t) = f(t) + \int_0^t K_1(s,t)X(s)ds + \int_\alpha^\beta K_2(s,t)X(s)ds, \quad t \in [0,T), \tag{1}$$

where $X(t)$ is the unknown function, $f(t)$ is analytic function, while $K_1(s,t)$ and$K_2(s,t)$ are the kernels of L^2 functions. In order to obtain an approximate solution for Eq. (1) based on modification of block pulse functions, we derive a new integration operational matrix and reduce our problem to solving a system of linear algebraic equations. Moreover, a new technique for computation of the linear terms in such equations is presented. Furthermore, convergence analysis of modification of block pulse functions is investigated. We also demonstrate the efficiency and accuracy of the proposed method.

2. Materials and Methods

2.1. Definition of Modified Block Pulse Functions. A set of ε modified block pulse functions (εMBPFs) $\psi_i(t), i = 0, 1, , m$ on the interval $[0,T)$ are defined as

$$\psi_0(t) = \begin{cases} 1, & t \in [0, h-\varepsilon) = I_0, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi_i(t) = \begin{cases} 1, & t \in [ih-\varepsilon, (i+1)h-\varepsilon) = I_i, \\ 0, & \text{otherwise}, \end{cases}$$

for $i = 1, 2, \ldots, m-1$ and

$$\psi_m(t) = \begin{cases} 1, & t \in [T-\varepsilon, T) = I_m, \\ 0, & \text{otherwise}, \end{cases}$$

with a positive integer value for m and $h = \frac{T}{m}$.

2.2. Properties of MBPFs. The important properties of εMBPFs are as follows

- Disjointness:

$$\psi_i(t)\psi_j(t) = \begin{cases} \psi_i(t), & i = j, \\ 0, & i \neq j, \end{cases}$$

  where $i, j = 0, \ldots, m$.
- Orthogonality:

$$\int_0^T \psi_i(t)\psi_j(t)dt = h\delta_{ij},$$

  where $i, j = 1, \ldots, m-1$ and δ_{ij} is Kronecker delta.
- Completeness:

$$\int_0^T f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \|\psi_i(t)\|^2,$$

  where

$$f_i = \frac{1}{\Delta(l_i)} \int_0^T f(t)\psi_i(t)dt, \tag{2}$$

and $\Delta(l_i)$ is length of interval I_i

2.3. **Function Approximation.** Rewriting Eq. (2) in the vector form we have

$$f(t) \simeq \sum_{i=0}^{m} f_i \psi_i(t) = F^T \Psi(t) = \Psi^T(t) F,$$

in which

$$F = (f_0 \ \ f_1 \ \ \dots \ \ f_m)^T$$

and

$$\Psi(t) = (\psi_0(t) \ \ \psi_1(t) \ \ \dots \ \ \psi_m(t))^T.$$

Moreover, any two dimensional function $k(s,t) \in L^2([0,T_1) \times [0,T_2))$ can be expanded with respect to εMBPFs such as

$$k(s,t) \simeq \Psi^T(s) K \Psi(t) = \Psi^T(t) K^T \Psi(s),$$

where $\Psi(s)$ and $\Psi(t)$ are m_1 and m_2 dimensional εMBPFs vectors respectively, and $K = (k_{ij}), i = 0,1,\dots,m_1, j = 0,1,\dots,m_2$ is the $m_1 \times m_2$ ε modified block pulse coefficient matrix with

$$k_{ij} = \frac{1}{\Delta(I_i)\Delta(I_j)} \int_0^{T_1} \int_0^{T_2} k(s,t) \Psi_i(s) \Psi_j(t) dt ds.$$

For convenience, we put $m_1 = m_2 = m$. We defining $\Psi_{m+1}(t) = (\Psi_0(t) \ \ \Psi_1(t) \ \ \dots \ \ \Psi_m(t))^T$, we have

$$\Psi_{m+1}(t) \Psi_{m+1}^T(t) = \begin{pmatrix} \psi_0(t) & 0 & \dots & 0 \\ 0 & \psi_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \psi_m(t) \end{pmatrix}_{(m+1)\times(m+1)}$$

Furthermore

$$\Psi_{m+1}^T(t) \Psi_{m+1}(t) = 1,$$

and

$$\Psi_{m+1}(t) \Psi_{m+1}^T(t) F = D_F \Psi_{m+1}(t),$$

where D_F usually denotes a diagonal matrix whose diagonal entries are related to a constant vector $F = (f_0 \ \ f_1 \ \ \dots \ \ f_m)^T$.

2.4. **Intergration Operational Matrix.** Similar to block pulse functions,

$$\int_0^t \Psi_{m+1}(s) ds \simeq Q \Psi_{m+1}(t),$$

where the integration operational matrix Q of εMBPFs is given by

$$Q = \begin{pmatrix} \frac{h-\varepsilon}{2} & h-\varepsilon & \dots & h-\varepsilon \\ 0 & \frac{h}{2} & \dots & h \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\varepsilon}{2} \end{pmatrix}_{(m+1)\times(m+1)}.$$

So, the integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s) ds \simeq \int_0^t F^T \Psi_{m+1}(s) ds \simeq F^T Q \Psi_{m+1}(t).$$

3. Result and Discussion

3.1. **Solving Volterra-Fredholm Integral Equations by Modification of Block Pulse Functions.** We consider following linear Volterra-Fredholm integral equation

$$X(t) = f(t) + \int_\alpha^\beta K_1(s,t)X(s)ds + \int_0^t K_2(s,t)X(s)ds, \quad t \in [0,T). \tag{3}$$

We approximate functions $X(t), f(t), k_1(s,t)$, and $k_2(s,t)$ by εMBPFs as follows

$$X(t) \simeq \Phi^T(t)W = W^T\Phi(t),$$

$$f(t) \simeq \Phi^T(t)F = F^T\Phi(t),$$

$$K_1(s,t) \simeq \Phi^T(s)K_1\Phi(t) = \Phi^T(t)K_1^T\Phi(s),$$

$$K_2(s,t) \simeq \Phi^T(s)K_2\Phi(t) = \Phi^T(t)K_2^T\Phi(s).$$

In the above approximation, W and F are modified block pulse coefficients vector, K_1 and K_2 are modified block pulse coefficients matrix.

Substituting above approximation in Eq. (3), we get

$$W^T\Phi(t) \simeq F^T\Phi(t) + W^T\left(\int_\alpha^\beta \Phi(s)\Phi^T(s)ds\right)K_1\Phi(t) + W^T\left(\int_0^t \Phi(s)\Phi^T(s)ds\right)K_2\Phi(t). \tag{4}$$

Let K_j^i be the ith row of the constant matrices $K_j, j = 1,2,3. R^i$ be the ith row of the integration operational matrix $Q, D_{K_j^i}$ be diagonal matrices with K_j^i as its diagonal entries. By the relation $\int_\alpha^\beta \Phi(s)\Phi^T(s)ds = hI_{(m_1+1)\times(m_2+1)}$ and assuming $m_1 = m_2 = m$, we have

$$\left(\int_\alpha^\beta \Phi(s)\Phi^T(s)ds\right)K_1\Phi(t) = hIK_1\Phi(t) = B_1\Phi(t), \tag{5}$$

where $B_1 = hIK_1 = hK_1$.

Furthermore,

$$\left(\int_0^t \Phi(s)\Phi^T(s)ds\right)K_2\Phi(t) = \begin{pmatrix} R^0\Phi(t) & 0 & \dots & 0 \\ 0 & R^1\Phi(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R^m\Phi(t) \end{pmatrix}\begin{pmatrix} K_2^0 \\ K_2^1 \\ \vdots \\ K_2^m \end{pmatrix}\Phi(t)$$

$$= \begin{pmatrix} R^0\Phi(t)K_2^0\Phi(t) \\ R^1\Phi(t)K_2^1\Phi(t) \\ \vdots \\ R^m\Phi(t)K_2^m\Phi(t) \end{pmatrix}$$

$$= \begin{pmatrix} R^0\Phi(t)\Phi(t)^T {K_2^0}^T \\ R^1\Phi(t)\Phi(t)^T {K_2^1}^T \\ \vdots \\ R^m\Phi(t)\Phi(t)^T {K_2^m}^T \end{pmatrix}$$

$$= \begin{pmatrix} R^0 D_{K_2^0} \\ R^1 D_{K_2^1} \\ \vdots \\ R^m D_{K_2^m} \end{pmatrix}\Phi(t) = B_2\Phi(t), \tag{6}$$

where

$$B_2\begin{pmatrix} k_{00}\left(\frac{h-\varepsilon}{2}\right) & k_{01}(h-\varepsilon) & \dots & k_{0m}h-\varepsilon \\ 0 & k_{11}\left(\frac{h}{2}\right) & \dots & k_{1m}(h) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_{mm}\left(\frac{\varepsilon}{2}\right) \end{pmatrix}_{(m+1)\times(m+1)},$$

with substituting (5) and (6) in (4), we get

$$W^T\Phi(t) \simeq F^T\Phi(t) + W^TB_1\Phi(t) + W^TB_2\Phi(t).$$

Then,

$$W^T(I - B_1 - B_2) \simeq F^T$$

So, by getting $N = (I - B_1 - B_2)^2$ and replacing $\simeq$ by $=$, we have

$$NW = F.$$

Which is a linear system of equations with upper triangular coefficients matrix that gives the approximate modified block pulse coefficient of the unknown $X(t)$.

3.2. **Error Analysis.** In the following theorems, for simplicity we assume $T = 1$ and $h = \frac{1}{m}$.

Theorem 3.1. *If $\hat{f}_m(t) = \sum_{i=0}^{m} f_i\psi_i(t)$ and $f_i = \frac{1}{\Delta(I_i)}\int_0^1 f(t)\psi_i(t)dt, i = 0,\dots,m$ then:*

i $\delta = \int_0^1 (f(t) - \sum_{i=0}^{m} f_i\psi_i(t))^2 dt$, achieves its minimum value.
ii $\{\hat{f}_m(t)\}$ approaches f(t) pointwise.
iii $\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2\|\psi_i\|^2$.

Proof. Proof is like similar theorem in [3] but intervals of integration have to redefine as $I_i, i = 0,\dots,m$ in (3.1)

Theorem 3.2. *Assume:*

i $f(t)$ is continuous and differentiable in $[-h, 1+h]$ with bounded derivative, that is $|f'(t)| < M$.
ii $\hat{f}_{\frac{ih}{k}}(t), i = 0,\dots,k-1$, are correspondingly BPFs, $\frac{h}{k}$MBPFs,..., $\frac{(k-1)h}{k}$ MBPFs expansions of $f(t)$ base on $m+1\varepsilon$MBPFs over interval $[0, 1)$.
iii $\bar{f}(t) = \frac{1}{k}\sum_{i=0}^{k-1}\hat{f}_{\frac{ih}{k}}(t)$.

Then

$$\left\|f(t) - \hat{f}_{\frac{ih}{k}}(t)\right\| = \mathcal{O}(h), \text{ and } \|f(t) - \bar{f}(t)\| = \mathcal{O}\left(\frac{h}{k}\right) \text{ in } [h, 1-h].$$

Proof. Trapezoidal rule for integral is

$$\int_a^b f(t)dt = \frac{b-a}{2}(f(a) + f(b)) - \frac{(b-a)^3 f''(\eta)}{12}$$
$$= \frac{b-a}{12}(f(a) + f(b)) + E, \quad \eta \in [a, b], \tag{7}$$

where E is error of integration. Suppose $t_i = \frac{i}{m} = ih$ and $I_i = [t_{i-1}, t_i]$. The representation error when $f(t)$ is represented by a series of BPFs over every subinterval $\left[t_i, t_i + \frac{h}{k}\right], i = 0,\dots,m-1$ is

$$e_i(t) = f(t) - f_i\psi_i(t) = f(t) - f_i,$$

where $f_i = \frac{1}{h}\int_{ih}^{(i+1)h} f(t)dt$. From (7),

$$f_i = \frac{1}{2}(f(t_i) + f(t_i + h)) + E.$$

It is obvious that if $f(t) = C$ (constant), then $e_i(t) = 0$. So, this error is computed for $f(t) = t$ in interval $[t_i, t_i + \frac{h}{k}], i = 1,\dots,m-1$.

For this function $E = 0$, so

$$e_i(t)_{\left[t_i, t_i+\frac{h}{k}\right]} = |t - f_i| = \left|t - \frac{t_i + t_{i+1}}{2}\right| = \left|t - \left(t_i + \frac{h}{2}\right)\right| \leq \frac{h}{2},$$

then this error with BPFs is $\frac{h}{2}M$.

Similarly, the error when $f(t)$ is represented in a series of εMBPFs over every subinterval $[t_i, t_i + \frac{h}{k}]$ is

$$\begin{aligned} e_i(t)_{[t_i, t_i+\frac{h}{k}]} &= \left|t - \left(\frac{\sum_{j=0}^{k-1}\left(t_i - \left(\frac{jh}{k}\right) + t_{i+1} - \left(\frac{jh}{k}\right)\right)}{2k}\right)\right| \\ &= \left|t - \left(\frac{\sum_{j=0}^{k-1}\left(t_i - \left(\frac{jh}{k}\right) + t_i + h - \left(\frac{jh}{k}\right)\right)}{2k}\right)\right| \\ &= \left|t - \left(t_i + \frac{h}{2}\right) - \frac{(k-1)h}{2k}\right| \\ &\leq \frac{h}{2k}. \end{aligned}$$

So, the error with εMBPFs is $\frac{h}{2k}M$.

For I_0 in $\left[0, \frac{h}{k}\right]$ we have

$$\begin{aligned} e_i(t)_{\left[0, \frac{h}{k}\right]} &= \left|t - \sum_{j=0}^{k-1} \frac{h - \left(\frac{jh}{k}\right)}{2k}\right| \\ &= \left|t - \left(\frac{h}{2} - \frac{(k-1)h}{4k}\right)\right| \\ &= \left|t - \left(\frac{h}{4} + \frac{h}{4k}\right)\right| \\ &= \mathcal{O}\left(\frac{h}{4}\right) \end{aligned}$$

So, the error is $\mathcal{O}\left(\frac{h}{4}\right)$ also for I_n

Now,

$$
\begin{aligned}
\|e_i(t)\|^2 &= \int_{t_i}^{t_i+\frac{h}{k}} |e_i(t)|^2 dt \\
&= \int_{t_i}^{t_i+\frac{h}{k}} \frac{h^2}{4k^2} M^2 dt \\
&= \frac{h^3}{4k^3} M^2, \\
\|e_i\|^2 &= \int_0^1 e^2(t) dt \\
&= \int_0^1 \left(\sum_{i=1}^{m} \sum_{j=0}^{k-1} e_i(t) \right)^2 dt \\
&= \sum_{i=1}^{m} \sum_{j=0}^{k-1} \int_0^1 e_i^2(t) dt \\
&= \sum_{i=1}^{m} \sum_{j=0}^{k-1} \|e_i(t)\|^2 \\
&= \frac{1}{h}.k.\frac{h^3}{4k^3} M^2 \\
&= \frac{h^2}{4k^2} M^2.
\end{aligned}
$$

We define the representation error between $f(s,t)$ and its 2D-εMBPFs expansion f_{ij} over every subregion D_{ij}, is defined as

$$e_{ij}(s,t) = f(s,t) - f_{ij},$$

where $D_{ij} := \left\{ (s,t) \,\middle|\, t_i \le s \le t_i + \frac{h}{k}, t_j \le t \le t_j + \frac{h}{k} \right\}$.

Based on Taylors expansion and similarity to the above discussion,

$$\|e(s,t)\| = \frac{h}{2k} M.$$

Theorem 3.3. *Assume that*

i $P(\omega \in \Omega : \|u(\omega,t)\| < C) = 1$
ii $\|k_i\| < C, i = 1, 2.$

Then

$$\sup_{0 \le t \le T} (E(\|u - \bar{u}\|)^2)^{\frac{1}{2}} = \mathcal{O}\left(\frac{h}{k}\right), \quad t \in [h, 1-h].$$

PROOF. For a complete proof see [6].

3.3. Examples of linear Volterra-Fredholm integral equations.

Example 3.4. *consider the linear Volterra-Fredholm integral equation*

$$f(t) = 1 - \int_0^t (t-s) f(s) ds + \int_0^{\pi} f(s) ds,$$

with the exact solution is $f(t) = \cos t$.

TABLE 1. The exact and approximate solution of Example 3.4 for $m = 257$

t	Exact	Approximation
0	1	1.004
0.3	0.9553	0.958
0.6	0.8254	0.8274
0.9	0.6216	0.6234
1.2	0.362	0.3641
1.5	0.07053	0.07265
1.8	-0.2268	-0.2256
2.1	-0.5046	-0.5035
2.4	-0.7373	-0.7365
2.7	-0.9043	-0.9049
3	-0.99	-0.9927

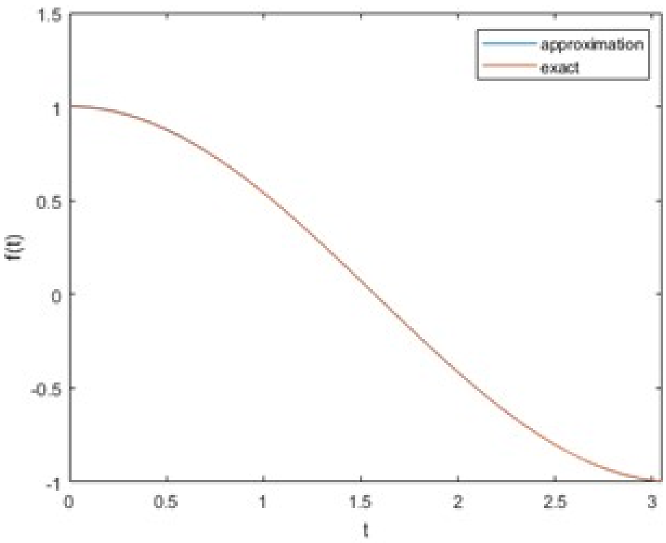

FIGURE 1. The trajectory of the exact and approximation solution of Example 3.4

Example 3.5. *consider the linear Volterra-Fredholm integral equation*

$$f(t) = t - 2e^t + e^{-t} + 1 + \int_0^t se^t f(s)ds + \int_0^1 e^{s+t} f(s)ds,$$

with the exact solution is $f(t) = e^{-t}$.

TABLE 2. The exact and approximate solution of Example 3.5 for $m = 257$

t	Exact	Approximation
0	1	0.9991
0.1	0.9048	0.9017
0.2	0.8187	0.8162
0.3	0.7408	0.7388
0.4	0.6704	0.6688
0.5	0.6065	0.6054
0.6	0.5488	0.548
0.7	0.4966	0.496
0.8	0.4493	0.449
0.9	0.4066	0.4064
1	0.3679	0.3683

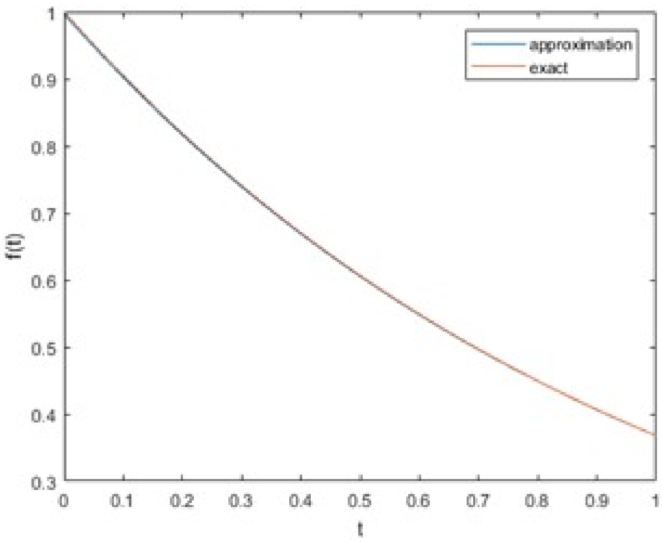

FIGURE 2. The trajectory of the exact and approximation solution of Example 3.5

4. CONCLUSIONS

The εMBPFs and their integration operational matrix are used to obtain the solution of linear Volterra-Fredholm integral equations. The present method reduces a linear Volterra-Fredholm integral equations into a system of algebraic equations. The convergence and error analysis of the proposed method are investigated. Some numerical examples are given, we plot approximate and exact solution to demonstrate the efficiency and accuracy of the proposed method. The results show that the approximate solutions of the proposed method have a good of efficiency and accuracy.

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