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Research Paper

Enumerating the Number of Connected Vertices Labeled Graph of Order Six with Maximum Ten Loops and Containing No Parallel Edges

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Abstract
A graph G (V, E) is said to be a connected graph if for every two vertices on the graph there exist at least a path connecting them, otherwise, the graph is disconnected. Two edges or more that connect the same pair of vertices are called parallel edges, and an edge that starts and ends at the same vertex is called a loop. A graph is called simple if that graph contains no loops nor parallel edges. Given n vertices and m edges, m ≥ 1, there are many graphs that can be formed, either connected or disconnected. In this research, we find that the formula to count the number of connected vertices labeled graphs of order six with m edges that containing maximum ten loops and no parallel edges $N(G(\ell)_{6,m}) = \sum_{t=5}^{15} N(G(\ell)_{6,m,t})$, where t ≤ m; and $N(G(\ell)_{6,m,5}) = 1296\binom{m}{5}$, $N(G(\ell)_{6,m,6}) = 1980\binom{m-1}{5}$, $N(G(\ell)_{6,m,7}) = 3330\binom{m-2}{5}$, $N(G(\ell)_{6,m,8}) = 4620\binom{m-3}{5}$, $N(G(\ell)_{6,m,9}) = 6660\binom{m-4}{5}$, $N(G(\ell)_{6,m,10}) = 2640\binom{m-5}{5}$, $N(G(\ell)_{6,m,11}) = 1155\binom{m-6}{5}$, $N(G(\ell)_{6,m,12}) = 420\binom{m-7}{5}$, $N(G(\ell)_{6,m,13}) = 150\binom{m-8}{5}$, $N(G(\ell)_{6,m,14}) = 15\binom{m-9}{5}$, $N(G(\ell)_{6,m,15}) = \binom{m-10}{5}$

Keywords
graph, disconnected, vertices, labeled, loops

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1. INTRODUCTION

There is no doubt that many real-life problems can be represented using graph theory. Many branches of science use graph theory applications include chemistry, biology, computer science, economics, engineering, and others. For example, in a transportation problem, the cities can be represented by vertices while the roads that connect the cities can be represented by edges. Moreover, the edge in the graph can be assigned a number that can represent non-structural information such as cost, time, distance, and others. By representing the transportation network into a graph, the situation can be easily visualized. Some applications of graph theory are given for example: in networks design (Hsu and Lin, 2008), in cryptography (Al Etaiwi (2014); Priyadarsini (2015)), in phylogenesis (Mathur and Adlakha (2016); Deka (2015); Brandes and Cornelsen (2009)), etc.

In 1847 G.R. Kirchhoff, in order to solve the linear equations that gave the current around each circuit and each branch of and electrical networks, developed the theory of tree (Vasudev, 2006). One historical work related to graph enumeration was done by Cayley in 1857 who enumerated the isomer of hydrocarbon C_nH_{2n+2} using the concept of tree (Cayley (1874)). Harary and Enumeration (1977) discussed the basic idea for graph enumeration. The method for enumerating trees and forest was given by Bona (2007), and the use of generating function for enumeration is given by Stanley (1997);Stanley (1999) and Agnarsson and Greenlaw (2006)); and introduction to various combinatorial counting techniques is given by Wilf (1994).

Graphically, given n vertices and m edges, there are many graphs that can be formed. The number of disconnected vertices labeled graphs of order five containing no parallel edges is investigated Wamiliana et al. (2016). In 2018, the formula for counting the number of disconnected vertices labeled graphs of order five with maximum six 3-parallel edges was observed Efendi et al. (2018), and in 2019 how to enumerate the number of connected vertices labeled graph of order five with maximum five parallel edges and containing no loops is discussed Wamiliana et al. (2019). In this article we will discuss the formula to enumerate the number of connected vertices labeled graphs of order six with maximum ten loops and containing no parallel edges.

The paper is organized as follows: after Introduction is given in Section 1, Construction and Pattern Obtained is discussed in Section II. Result and Discussion are given in Section III, follows by Conclusion in Section IV.

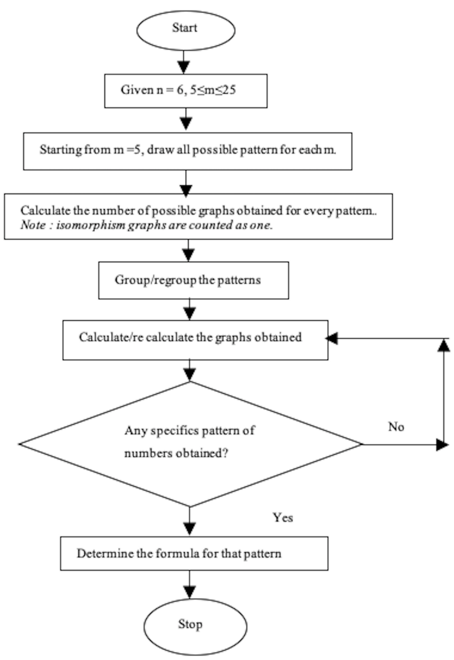

Figure 1. The procedure for finding the formula

2. OBSERVATION AND INVESTIGATION

The following diagram shows the procedure to find the formula to enumerate the number of connected vertices labeled graphs of order six with maximum ten loops and containing no parallel edges.

Given n vertices, n = 6, and m the number of edges, 5 ≤ m ≤ 25, there are lot of graphs that can be formed. In this study, we construct connected vertex labeled graphs using n and m given. Moreover, the graphs constructed contain no parallel edges and may contain at most ten loops. Note that isomorphism graphs are counted as one. There are many graphs that are able to be obtained, but in this paper we only provide some patterns of graphs that can be obtained due to space limitation.

Those graphs in the first row of Figure 2 are some connected graphs that can be formed if we are given n = 6 and m = 5, and in the second row are examples of connected graphs that can be created if we are given n=6 and m = 6, 7, 8, 9. Note that the graph is vertices labeled. Therefore, the position of the labeled is taken into account. However, isomorphic graphs are counted as one. For example, if we interchange label v1 to v6 in the left-most first-row picture in Figure 2, we have different graphs.

3. RESULTS AND DISCUSSION

As already stated in Section 2, in flow diagram for finding the formula, we start by observing the simplest form of the graph under consideration (n=6, m=5 and t=5), where m is the total number of edges, t is the number of edges that connect different pairs of vertices. Then, we construct all possible patterns according to that requirement. The four graphs on the first row in Figure 2 are some patterns that are able to be created according to this

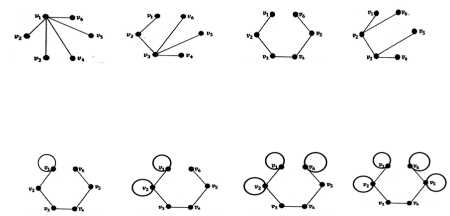

Figure 2. Some patterns that can be obtained

condition. Note that the graphs that are constructed must be connected graphs. Moreover, if two or more graphs created are isomorphic then those graphs are counted as one, for example, if we interchange v6 and v2 in the left-most first row in Figure 2, then those two graphs are isomorphic and counted as one. After the observation process, the next step is calculating the number of graphs for every pattern. For example, for the pattern in the left-most first-row picture in Figure 2, we only get six graphs. Next step is grouping the graphs created. By grouping the graphs obtained in term of m and t (loops are not contributing to t), the number of graphs obtained can be put in the following table:

The numbers in the table show the numbers of disconnected graphs created according to given m and t. From Table ??, we can see that the number in every column constitute a pattern. Note that in every column there is a sequence of numbers (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003) appears as follows:
t=5: 1296 (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003)
t=6: 1980 (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003)
t=7: 3330 (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003)
t=8: 4620 (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003)
t=9: 6660 (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003)
t=10: 2460 (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003)
t=11: 1155 (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003)
t=12: 420 (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003)
t=13: 150 (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003)
t=14: 15 (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003)
4=15: 1 (1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003)

Notate $G(\ell)_{6,m,t}$ as connected graph of order six with m edges and t number of edges that connect different pair of vertices and contains no parallel edges (loops allowable), $N(G(\ell)_{6,m,t})$ as the number of graphs of order six with m edges and t number of edges that connect different pair of vertices and contains no parallel edges (loops allowable).

For t=5: the sequence 1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003 is observed.

1 6 21 56 126 252 462 792 1287 2002 3003
5 15 35 70 126 210 330 495 715 1001
10 20 35 56 84 120 165 220 286
10 15 21 28 36 45 55 66
5 6 7 8 9 10 11
1 1 1 1 1 1

Table 1. The number of connected vertices labeled graph of order six with a maximum ten loops and containing no parallel edges.

m	The number of connected vertices labeled graphs of order six with maximum ten loops without parallel edges.										
	t										
	5	6	7	8	9	10	11	12	13	14	15
5	1296										
6	7776	1980									
7	27216	11880	3330								
8	72576	41580	19980	4620							
9	163296	110880	69930	27720	6660						
10	326592	249480	186480	97020	39960	2460					
11	598752	498960	419580	258720	139860	14760	1155				
12	1026432	914760	839160	582120	372960	51660	6930	420			
13	1667952	1568160	1538460	1164240	839160	137760	24255	2520	150		
14	2594592	2548260	2637360	2134440	1678320	309960	64680	8820	900	15	
15	3891888	3963960	4285710	3659040	3076920	619920	145530	23520	3150	90	1
16		5945940	6666660	5945940	5274720	1136520	291060	52920	8400	315	6
17			9999990	9249240	8571420	1948320	533610	105840	18900	840	21
18				13873860	13333320	3166020	914760	194040	37800	1890	56
19					19999980	4924920	1486485	332640	69300	3780	126
20						7387380	2312310	540540	118800	6930	252
21							3468465	840840	193050	11880	462
22								1261260	300300	19305	792
23									450450	30030	1287
24										45045	2002
25											3003

By observing the numbers in every row, Table 1 can be put in the following table:

The first sequence of numbers shows the numbers that appear in the first column that multiply by 1296. The second sequence shows the difference of two consecutive numbers in the first sequence, while the third sequence shows the difference of two consecutive numbers in the second sequence, and so on, until the last sequence which is formed a fixed difference.

Result 1: Given n = 6, 5 ≤ m ≤ 25, t = 5, the number of connected graphs of order six with m edges and t number of edges that connect different pair of vertices and contains no parallel edges (loops allowable) is $N(G(\ell)_{6,m,5}) = 1296\binom{m}{5}$

Proof: From the sequence of numbers above we can see that the fixed differences occur on the fifth level. Thus the sequence can be represented by polynomial of order five:
$Q_5(m) = A_5m^5 + A_4m^4 + A_3m^3 + A_2m^2 + A_1m + A_0$
By Substituting m = 5, 6, 7, 8, 9, 10 to the polynomial we get the following system of equations:

$$1296 = 3125A_5 + 625A_4 + 125A_3 + 25A_2 + 5A_1 + A_0 \quad (1)$$

$$7776 = 7776A_5 + 1296A_5 + 216A_5 + 36A_5 + 6A_5 + A_5 \quad (2)$$

$$27216 = 16807A_5 + 2401A_5 + 343A_5 + 49A_5 + 7A_5 + A_5 \quad (3)$$

$$72576 = 32768A_5 + 4096A_5 + 512A_5 + 64A_5 + 8A_5 + A_5 \quad (4)$$

$$163296 = 59049A_5 + 6561A_5 + 729A_5 + 81A_5 + 9A_5 + A_5 \quad (5)$$

$$326592 = 100000A_5 + 10000A_5 + 1000A_5 + 100A_5 + 10A_5 + A_5 \quad (6)$$

By solving that system of linear equations we get $A_5 = \frac{54}{5}, A_4 = -108, A_3 = 378, A_2 = -540, A_1 = \frac{-1296}{5}$ and $A_0 = 0$

Therefore

$$Q_5(m) = \frac{54}{5}m^5 - 108m^4 + 378m^3 - 540m^2 + \frac{1296}{5}$$

$$= \frac{54}{5}m(m^4 - 10m^3 + 35m^2 - 50m + 24)$$

$$= \frac{54}{5}m(m-1)(m-2)(m-3)(m-4)$$

Table 2. Another form of Table 1

The number of connected vertices labeled graphs of order six with maximum ten loops.											
m	*t*										
	5	6	7	8	9	10	11	12	13	14	15
5	1 x 1296										T
6	6 x 1296	1 x 1980									
7	21 x 1296	6 x 1980	1 x 3330								
8	56 x 1296	21 x 1980	6 x 3330	1 x 4620							
9	126 x 1296	56 x 1980	21 x 3330	6 x 4620	1 x 6660						
10	252 x 1296	126 x 1980	56 x 3330	21 x 4620	6 x 6660	1 x 2460					
11	462 x 1296	252 x 1980	126 x 3330	56 x 4620	21 x 6660	6 x 2460	1 x 1155				
12	792 x 1296	462 x 1980	252 x 3330	126 x 4620	56 x 6660	21 x 2460	6 x 1155	1 x 420			
13	1287 x 1296	792 x 1980	462 x 3330	252 x 4620	126 x 6660	56 x 2460	21 x 1155	6 x 420	1 x 150		
14	2002 x 1296	1287 x 1980	792 x 3330	462 x 4620	252 x 6660	126 x 2460	56 x 1155	21 x 420	6 x 150	1 x 15	
15	3003 x 1296	2002 x 1980	1287 x 3330	792 x 4620	462 x 6660	252 x 2460	126 x 1155	56 x 420	21 x 150	6 x 15	1 x1
16		3003 x 1980	2002 x 3330	1287 x 4620	792 x 6660	462 x 2460	252 x 1155	126 x 420	56 x 150	21 x 15	6 x1
17			3003 x 3330	2002 x 4620	1287 x 6660	792 x 2460	462 x 1155	252 x 420	126 x 150	56 x 15	21 x1
18				3003 x 4620	2002 x 6660	1287 x 2460	792 x 1155	462 x 420	252 x 150	126 x 15	56 x1
19					3003 x 6660	2002 x 2460	1287 x 1155	792 x 420	462 x 150	252 x 15	126 x1
20						3003 x 2460	2002 x 1155	1287 x 420	792 x 150	462 x 15	252 x1
21							3003 x 1155	2002 x 420	1287 x 150	792 x 15	462 x1
22								3003 x 420	2002 x 150	1287 x 15	792 x1
23									3003 x 150	2002 x 15	1287 x1
24										3003 x 15	2002 x1
25											3003 x1

$$= \frac{\frac{54}{5}.5.4.3.2.1m(m-1)(m-2)(m-3)(m-4)(m-5)!}{(5.4.3.2.1.(m-5)!}$$

$$= 1296x\binom{m}{5}$$

For t=6: the sequence 1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003 is observed

1 6 21 56 126 252 462 792 1287 2002 3003

5 15 35 70 126 210 330 495 715 1001

10 20 35 56 84 120 165 220 286

10 15 21 28 36 45 55 66

5 6 7 8 9 10 11

1 1 1 1 1 1

Result 2: Given n=6, 6≤m≤25, t=6, the number of connected graphs of order six with m edges and t number of edges that connect different pair of vertices and contains no parallel edges (loops allowable), $N(G(\ell)_{6,m,6})$ is 1980 × $C_5^{(m-1)}$

Proof: From the sequence of numbers above we can see that the fixed differences occurs on the fifth level. Thus the sequence can be represented by polynomial of order five: $Q_5(m) = A_5 m5 + A_4 m4 + A_3 m3 + A_2 m2 + A_1 m + A_0$

By substituting m= 6,7,8,9,10,11 to the polynomial we get the following system of equations:

$1980 = 7776A_5 + 1296A_4 + 216A_3 + 36A_2 + 6A_1 + A_0$

$11880 = 16807A_5 + 2401A_4 + 343A_3 + 49A_2 + 7A_1 + A_0$

$41580 = 32768A_5 + 4096A_4 + 512A_3 + 64A_2 + 8A_1 + A_0$

$110880 = 59049A_5 + 6561A_4 + 729A_3 + 81A_2 + 9A_1 + A_0$

$249480 = 100000A_5 + 10000A_4 + 1000A_3 + 100A_2 + 10A_1 + A_0$

$498960 = 161051A_5 + 14641A_4 + 1331A_3 + 121A_2 + 11A_1 + A_0$

By solving that system of linear equations we get $A5 = \frac{32}{2}$, $A4 = \frac{(-495)}{2}$, $A3 = \frac{2805}{2}$, $A2 = \frac{(-7425)}{2}$, A1 = 4521, A0= -1980

Therefore

$$Q_5(m) = \frac{33}{2}m_5 + \frac{(-495)}{2}m_4 + \frac{2805}{2}, m_3 + \frac{(-7425)}{2} + 4521m - 1980$$

$$= \frac{\frac{165}{2}}{5}(m^5 - 15m^4 + 85m^3 - 225m^2 + 274m + 120)$$

$$= \frac{\frac{165}{2}}{5}(m-1)(m-2)(m-3)(m-4)(m-5)$$

$$= 1980xC\binom{m-1}{5}$$

Continuing observation and proof with similar manner we get the following results:

a. for t = 7, $N(G(\ell)_{6,m,7}) = 3330\binom{m-2}{5}$
b. for t = 8, $N(G(l)6, m, 8) = 4620\binom{m-3}{5}$
c. for t = 9, $N(G(l)6, m, 9) = 6660\binom{m-4}{5}$
d. for t = 10, $N(G(l)6, m, 10) = 2640\binom{m-5}{5}$
e. for t = 11 $N(G(l)6, m, 11) = 1155\binom{m-6}{5}$
f. for t = 12, $N(G(l)6, m, 12) = 420\binom{m-7}{5}$
g. for t = 13, $N(G(l)6, m, 13) = 150\binom{m-8}{5}$
h. for t = 14, $N(G(l)6, m, 14) = 15\binom{m-9}{5}$
i. for t = 15, $N(G(l)6, m, 15) = \binom{m-10}{5}$

4. CONCLUSIONS

From the discussion above we can conclude that if there is given n vertices (n=6), m edges (5≤m≤25), and t (t is the number of edges that connect different pairs of vertices), then we can construct many connected graphs G(V,E), where n=|V|and m=|E|. Among those graphs obtained, there are graphs that may contained loops maximum ten. The formula for counting the number of connected graphs that contained maximum ten loops are as follow: Given n = 6,5≤ m ≤ 25, $N(G(\ell)6, m, t)$ the number of connected graphs of order six with m edges that containing maximum ten loops and no parallel edges $N(G(\ell)_{6,m})$ is: $N(G(\ell)_{6,m})$ $= \sum_{t=5}^{15} N(G(\ell)_{6,m,t})$, where t ≤ m; and $N(G(\ell)_{6,m,5})$
$= 1296\binom{m}{5}$, $N(G(\ell)_{6,m,6})$
$= 1980\binom{m-1}{5}$, $N(G(\ell)_{6,m,7})$
$= 3330\binom{m-2}{5}$, $N(G(\ell)_{6,m,8})$
$= 4620\binom{m-3}{5}$, $N(G(\ell)_{6,m,9})$
$= 6660\binom{m-4}{5}$, $N(G(\ell)_{6,m,10})$
$= 2640\binom{m-5}{5}$, $N(G(\ell)_{6,m,11}) = 1155\binom{m-6}{5}$, $N(G(\ell)_{6,m,12})$
$= 420\binom{m-7}{5}$, $N(G(\ell)_{6,m,13}) = 150\binom{m-8}{5}$,
$N(G(\ell)_{6,m,14}) = 15\binom{m-9}{5}$, $N(G(\ell)_{6,m,15}) = \binom{m-10}{5}$

5. ACKNOWLEDGEMENT

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REFERENCES

Agnarsson, G. and R. Greenlaw (2006). *Graph theory: Modeling, applications, and algorithms.* Prentice-Hall, Inc.

Al Etaiwi, W. M. (2014). Encryption algorithm using graph theory. *Journal of Scientific Research and Reports*; 2519–2527

Bona, M. (2007). *Introduction to enumerative combinatorics.* McGraw-Hill Science/Engineering/Math

Brandes, U. and S. Cornelsen (2009). Phylogenetic graph models beyond trees. *Discrete Applied Mathematics*, **157**(10); 2361–2369

Cayley, P. (1874). LVII. On the mathematical theory of isomers. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, **47**(314); 444–447

Deka, D. K. (2015). Application of Graph Theory in Phylogenetics: The Primate Approach. *Asia Pacific Mathematics Newsletter*

Efendi, M. F. N. et al. (2018). Penentuan Banyaknya Graf-Graf Tak Terhubung Berlabel Titik Berorde Lima Tanpa Loop Serta Banyaknya Garis 3-Paralel Maksimal Enam

Harary, F. and E. P. G. Enumeration (1977). Academic Press New York and London 1973. *Russian translation Moscow*

Hsu, L.-H. and C.-K. Lin (2008). *Graph theory and interconnection networks.* CRC press

Mathur, R. and N. Adlakha (2016). A graph theoretic model for prediction of reticulation events and phylogenetic networks for DNA sequences. *Egyptian Journal of Basic and Applied Sciences*, **3**(3); 263–271

Priyadarsini, P. (2015). A survey on some applications of graph theory in cryptography. *Journal of Discrete Mathematical Sciences and Cryptography*, **18**(3); 209–217

Stanley, R. (1997). Enumerative Combinatorics, 1, no. 49 of Cambridge Studies in Advanced Mathematics

Stanley, R. (1999). Enumerative Combinatorics, 2 no. 62 of Cambridge Studies in AdvancedMathematics. *Cambridge University Press, New York, 1999*

Vasudev, C. (2006). *Graph theory with applications.* New Age International

Wamiliana, A. Nuryaman, Amanto, A. Sutrisno, and N. A. Prayoga (2019). Determining the Number of Connected Vertices Labelled Graph of Order Five with Maximum Number of Parallel Edges is Five and Containing No Loops. *Journal of Physics: Conference Series*, **1338**; 012043

Wamiliana, W., A. Amanto, and G. T. Nagari (2016). Counting the Number of Disconnected Labeled Graphs of Order Five without Paralel Edges. *Internasional Series on Interdisciplinary Research*, **1**(1); 4–7

Wilf, ., H (1994). Generating Functionology. *Academic Press, second Edition, New York*