PENDUGAAN PARAMETER DERET WAKTU *HIDDEN* MARKOV SATU WAKTU SEBELUMNYA

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ABSTRAK. Pendugaan parameter deret waktu *Hidden* Markov satu waktu sebelumnya dilakukan mengunakan Metode *Maximum Likelihood* dan pendugaan ulang menggunakan metode *Expectation Maximization*. Dari kajian ini diperoleh algoritme untuk menduga parameter model.

Kata kunci: Rantai Markov, *Hidden* Markov, Deret waktu *Hidden* Markov, Metode *Expectation Maximization*.

1. PENDAHULUAN

Tulisan ini merupakan kajian tentang pendugaan parameter untuk deret waktu *Hidden* Markov satu waktu sebelumnya. Pendugaan parameter menggunakan metode *Maximum Likelihood* dan pendugaan ulangnya menggunakan metode *Expectatition Maximization* (Metode EM). Dari kedua metode tersebut kemudian diturunkan suatu algoritme yang dapat dipakai secara umum untuk menduga parameter model deret waktu *Hidden* Markov satu waktu sebelumnya.

Tulisan ini dimulai dengan definisi model deret waktu *Hidden* Markov satu waktu sebelumnya beserta sifat-sifatnya. Pada bagian 3 dibahas Pendugaan Parameter model dan terakhir pada bagian 4 dibahas pendugaan ulang parameter dan algoritmenya.

2. MODEL DERET WAKTU *HIDDEN* MARKOV SATU WAKTU SEBELUMNYA

Pasangan proses stokastik $\{(X_k, Y_k) : k \in \aleph\}$ yang terdefinisi pada ruang peluang $(\Omega, \mathcal{F}, P)$ dan mempunyai nilai pada $S \times \mathcal{Y}$ disebut model *hidden* Markov apabila $\{X_k\}$ adalah rantai Markov dengan *state* berhingga dan diasumsikan bahwa rantai Markov $\{X_k\}$ tidak diamati. Sehingga $\{X_k\}$ tersembunyi (*hidden*) di balik proses

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observasi $\{Y_k\}$. Banyaknya elemen dari S disebut ukuran (orde) dari model *hidden* Markov.

Pada bagian ini dibahas model *hidden* Markov yang merupakan deret waktu yang mempertimbangkan satu waktu sebelumnya dan berbentuk:

$$Y_t - \mu(X_t^*) = \phi(Y_{t-1} - \mu(X_{t-1}^*)) + \varepsilon_t \quad (2.1)$$

di mana:

- $\{\varepsilon_t\}$ adalah barisan peubah acak yang saling bebas dan menyebar normal $N(0, \sigma^2)$.
- $\{Y_t\}$ adalah proses yang diamati dan bernilai skalar
- $\{X_t^*\}$ adalah rantai Markov dengan ruang *state* $S^* = \{1, 2\}$ dan $P^* = \begin{bmatrix} p_{11}^* & p_{12}^* \\ p_{21}^* & p_{22}^* \end{bmatrix}$ merupakan matriks peluang transisinya, dengan $p_{ji}^* = P(X_t^* = j \mid X_{t-1}^* = i)$
- $\mu(X_t^*) = \langle \mu, X_t^* \rangle = \mu_{X_t^*}$, dengan $\langle .,. \rangle$ menyatakan hasil kali dalam di R^N.
- $\phi,\ \mu_1$ dan μ_2 adalah konstanta real.

Perhatikan bahwa model ini dicirikan oleh parameter $\theta = (\mu_1, \mu_2, \phi, \sigma^2, P)$. Dengan menggunakan metode EM akan diduga parameter $\theta = (\mu_1, \mu_2, \phi, \sigma^2, P)$ dari data Y.

Dalam kasus ini Y_t tidak hanya bergantung pada X_t^* tetapi juga bergantung pada X_{t-1}^* sehingga agar tetap memenuhi sifat Markov perlu didefinisikan proses baru $\{X_t\}$ di mana

$$\begin{aligned} X_t &= 1 \text{ jika } X_t^* = 1 \text{ dan } X_{t-1}^* = 1 \\ X_t &= 2 \text{ jika } X_t^* = 2 \text{ dan } X_{t-1}^* = 1 \\ X_t &= 3 \text{ jika } X_t^* = 1 \text{ dan } X_{t-1}^* = 2 \\ X_t &= 4 \text{ jika } X_t^* = 2 \text{ dan } X_{t-1}^* = 2. \end{aligned} \quad (2.2)$$

Lemma 2.1:
$\{X_t\}$ merupakan rantai Markov dengan matriks peluang transisi adalah

$$\mathbf{P} = \begin{bmatrix} p_{11}^* & 0 & p_{11}^* & 0 \\ p_{21}^* & 0 & p_{21}^* & 0 \\ 0 & p_{12}^* & 0 & p_{12}^* \\ 0 & p_{22}^* & 0 & p_{22}^* \end{bmatrix}. \quad (2.3)$$

Bukti: Lihat Setiawaty (2007)

Selanjutnya karena $\varepsilon_t \sim N\left(0,\sigma^2\right)$ bebas stokastik identik maka dapat diperoleh fungsi sebaran bagi ε_t sebagai berikut.

$$F_{\varepsilon_t}\left(y_t\right) = P\left(\varepsilon_t \le y_t\right) = \int_0^{y_t} \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{\frac{-\left(\varepsilon_t - 0\right)^2}{2\sigma^2}\right\} d\varepsilon_t = \int_0^{y_t} \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{\frac{-\left(\varepsilon_t\right)^2}{2\sigma^2}\right\} d\varepsilon_t \,. \tag{2.4}$$

Berdasarkan persamaan (2.4) diperoleh fungsi sebaran bagi Y_t

$$F_{Y_t}\left(y_t\right) = P\left(Y_t \le y_t\right) = P\left(\phi\left(y_{t-1} - \mu_{S_{t-1}^*}\right) + \mu_{S_t^*} + \varepsilon_t \le y_t\right) = P\left(\varepsilon_t \le \left(y_t - \mu_{S_t^*}\right) - \phi\left(y_{t-1} - \mu_{S_{t-1}^*}\right)\right) \cdot$$

Misalkan $v = \left(y_t - \mu_{S_t^*}\right) - \phi\left(y_{t-1} - \mu_{S_{t-1}^*}\right)$, maka

$$F_{Y_t}\left(y_t\right) = \int_0^{v} \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{\frac{-\left(\varepsilon_t\right)^2}{2\sigma^2}\right\} d\varepsilon_t$$

dan

$$f_{Y_t}\left(y_t\right) = \frac{\partial}{\partial y_t} F_{Y_t}\left(y_t\right) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{\frac{-\left(v\right)^2}{2\sigma^2}\right\} \cdot \frac{\partial v}{\partial y_t} = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{\frac{-\left(\left(y_t - \mu_{S_t}^*\right) - \phi\left(y_{t-1} - \mu_{S_{t-1}}^*\right)\right)^2}{2\sigma^2}\right\} \tag{2.5}$$

Misalkan $\{\mathrm{Y}_t\}$ adalah medan-σ yang dibangun oleh $\{Y_0, Y_1, ..., Y_t\}$. Karena $\{X_t\}$ merupakan rantai Markov 4 *state* maka terdapat 4 fungsi kerapatan peluang bagi $\{Y_t\}$. Kumpulan fungsi kerapatan peluang tersebut dilambangkan dengan η_t.

$$\eta_t = \begin{bmatrix} f\left(y_t \,|\, X_t = 1, \mathrm{Y}_{t-1}; \theta\right) \\ f\left(y_t \,|\, X_t = 2, \mathrm{Y}_{t-1}; \theta\right) \\ f\left(y_t \,|\, X_t = 3, \mathrm{Y}_{t-1}; \theta\right) \\ f\left(y_t \,|\, X_t = 4, \mathrm{Y}_{t-1}; \theta\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{\frac{-\left(\left(y_t - \mu_1\right) - \phi\left(y_{t-1} - \mu_1\right)\right)^2}{2\sigma^2}\right\} \\ \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{\frac{-\left(\left(y_t - \mu_2\right) - \phi\left(y_{t-1} - \mu_1\right)\right)^2}{2\sigma^2}\right\} \\ \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{\frac{-\left(\left(y_t - \mu_1\right) - \phi\left(y_{t-1} - \mu_2\right)\right)^2}{2\sigma^2}\right\} \\ \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{\frac{-\left(\left(y_t - \mu_2\right) - \phi\left(y_{t-1} - \mu_2\right)\right)^2}{2\sigma^2}\right\} \end{bmatrix} \tag{2.6}$$

Misalkan $\zeta_{t|t-1} = (\zeta_{t|t-1}^{(1)}, \zeta_{t|t-1}^{(2)}, \zeta_{t|t-1}^{(3)}, \zeta_{t|t-1}^{(4)})^T$ melambangkan vektor (4×1) di mana $\zeta_{t|t-1}^{(j)} = P(X_t = j \,|\, \mathrm{Y}_{t-1})$ dan $\otimes$ melambangkan perkalian elemen per-elemen dari dua vektor, maka

$$\hat{\xi}_{t|t-1} \otimes \eta_t = \begin{bmatrix} P\{X_t = 1|\text{Y}_{t-1};\theta\} \\ P\{X_t = 2|\text{Y}_{t-1};\theta\} \\ P\{X_t = 3|\text{Y}_{t-1};\theta\} \\ P\{X_t = 4|\text{Y}_{t-1};\theta\} \end{bmatrix} \otimes \begin{bmatrix} f\left(y_t|X_t = 1, \text{Y}_{t-1};\theta\right) \\ f\left(y_t|X_t = 2, \text{Y}_{t-1};\theta\right) \\ f\left(y_t|X_t = 3, \text{Y}_{t-1};\theta\right) \\ f\left(y_t|X_t = 4, \text{Y}_{t-1};\theta\right) \end{bmatrix} = \begin{bmatrix} P\left(X_t = 1|\text{Y}_{t-1};\theta\right) \cdot f\left(y_t|X_t = 1, \text{Y}_{t-1};\theta\right) \\ P\left(X_t = 2|\text{Y}_{t-1};\theta\right) \cdot f\left(y_t|X_t = 2, \text{Y}_{t-1};\theta\right) \\ P\left(X_t = 3|\text{Y}_{t-1};\theta\right) \cdot f\left(y_t|X_t = 3, \text{Y}_{t-1};\theta\right) \\ P\left(X_t = 41|\text{Y}_{t-1};\theta\right) \cdot f\left(y_t|X_t = 4, \text{Y}_{t-1};\theta\right) \end{bmatrix} \quad (2.7)$$

Misalkan $1' = (1,1,1,1)'$, maka

$$\begin{aligned} f\left(y_t|\text{Y}_{t-1};\theta\right) &= \mathbf{1}'\left(\hat{\xi}_{t|t-1} \otimes \eta_t\right) = \sum_{j=1}^{4} P\left\{X_t = j|\text{Y}_{t-1};\theta\right\} \cdot f\left\{y_t|X_t = j, \text{Y}_{t-1};\theta\right\} \\ &= P\left(X_t = 1|\text{Y}_{t-1};\theta\right)\frac{1}{\sqrt{2\pi\sigma}}\exp\left\{\frac{-\left((y_t - \mu_1) - \phi(y_{t-1} - \mu_1)\right)^2}{2\sigma^2}\right\} \\ &+ P\left(X_t = 2|\text{Y}_{t-1};\theta\right)\frac{1}{\sqrt{2\pi\sigma}}\exp\left\{\frac{-\left((y_t - \mu_2) - \phi(y_{t-1} - \mu_1)\right)^2}{2\sigma^2}\right\} \\ &+ P\left(X_t = 3|\text{Y}_{t-1};\theta\right)\frac{1}{\sqrt{2\pi\sigma}}\exp\left\{\frac{-\left((y_t - \mu_1) - \phi(y_{t-1} - \mu_2)\right)^2}{2\sigma^2}\right\} \\ &+ P\left(X_t = 4|\text{Y}_{t-1};\theta\right)\frac{1}{\sqrt{2\pi\sigma}}\exp\left\{\frac{-\left((y_t - \mu_2) - \phi(y_{t-1} - \mu_2)\right)^2}{2\sigma^2}\right\} \end{aligned} \quad (2.8)$$

Salah satu pendekatan yang dapat digunakan untuk memilih nilai awal bagi $\hat{\xi}_{t|t-1}$ adalah dengan membuat $\hat{\xi}_{1|0}$ sama dengan vektor dari peluang tak bersyarat $\pi = \left[\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4\right]'$ yang memenuhi sifat *ergodic*, yaitu

$$\pi = \mathbf{P}\pi$$

dan

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1.$$

Menurut Setiawaty (2007) diperoleh:

$$\hat{\xi}_{1|0} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} \dfrac{-p_{12}^* p_{11}^*}{p_{11}^* - p_{12}^* - 1} \\ \dfrac{p_{11}^* - 1 - p_{11}^* p_{22}^* + p_{22}^*}{p_{11}^* - p_{12}^* - 1} \\ \dfrac{p_{12}^*\left(p_{11}^* - 1\right)}{p_{11}^* - p_{12}^* - 1} \\ \dfrac{\left(p_{11}^* - 1\right)p_{22}^*}{p_{11}^* - p_{12}^* - 1} \end{bmatrix}. \quad (2.9)$$

Perhatikan bahwa

$$P(X_t = j \mid \text{Y}_t, \theta) = \frac{P(y_t, X_t = j \mid \text{Y}_{t-1}, \theta)}{f(y_t \mid \text{Y}_{t-1}, \theta)}$$

sehingga $\hat{\xi}_{t|t} = \dfrac{\hat{\xi}_{t|t-1} \otimes \eta_t}{1'\left(\hat{\xi}_{t|t-1} \otimes \eta_t\right)}$, selain itu $\hat{\xi}_{t+1|t} = P\hat{\xi}_{t|t}$ dan $\hat{\xi}_{t+m|t} = P^m\hat{\xi}_{t|t}$.

3. PENDUGAAN PARAMETER

Penduga kemungkinan maksimum bagi θ diperoleh dengan memaksimumkan fungsi log *likelihood*

$$\mathrm{L}(\theta) = \sum_{t=1}^{T} \log f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right).$$

Dengan membuat turunan pertama dari log *likelihood* terhadap parameter θ sama dengan nol maka diperoleh penduga parameter θ sebagai berikut.

Misalkan

$$B = \frac{1}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)} P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\hat{\theta}\right)$$

$$C = \frac{1}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)} P\left(X_t = 2 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 2, \mathrm{Y}_{t-1};\hat{\theta}\right)$$

$$D = \frac{1}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)} P\left(X_t = 3 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 3, \mathrm{Y}_{t-1};\hat{\theta}\right)$$

$$E = \frac{1}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)} P\left(X_t = 4 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 4, \mathrm{Y}_{t-1};\hat{\theta}\right) \qquad (3.1)$$

Berdasarkan persaman (2.8) diperoleh

$$\begin{aligned}
\frac{\partial f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}{\partial \mu_1} &= P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-\left(\left(y_t - \mu_1\right) - \phi\left(y_{t-1} - \mu_1\right)\right)^2}{2\sigma^2}\right\} \frac{(-2)\left(\left(y_t - \mu_1\right) - \phi\left(y_{t-1} - \mu_1\right)\right)(-1+\phi)}{2\sigma^2} \\
&+ P\left(X_t = 2 \middle| \mathrm{Y}_{t-1};\theta\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-\left(\left(y_t - \mu_2\right) - \phi\left(y_{t-1} - \mu_1\right)\right)^2}{2\sigma^2}\right\} \frac{(-2)\left(\left(y_t - \mu_2\right) - \phi\left(y_{t-1} - \mu_1\right)\right)(\phi)}{2\sigma^2} \\
&+ P\left(X_t = 3 \middle| \mathrm{Y}_{t-1};\theta\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-\left(\left(y_t - \mu_1\right) - \phi\left(y_{t-1} - \mu_2\right)\right)^2}{2\sigma^2}\right\} \frac{(-2)\left(\left(y_t - \mu_1\right) - \phi\left(y_{t-1} - \mu_2\right)\right)(-1)}{2\sigma^2} \\
&= P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right) \frac{\left(\left(y_t - \mu_1\right) - \phi\left(y_{t-1} - \mu_1\right)\right)(1-\phi)}{\sigma^2} \\
&- P\left(X_t = 2 \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = 2, \mathrm{Y}_{t-1};\theta\right) \frac{\left(\left(y_t - \mu_2\right) - \phi\left(y_{t-1} - \mu_1\right)\right)(\phi)}{\sigma^2} \\
&+ P\left(X_t = 3 \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = 3, \mathrm{Y}_{t-1};\theta\right) \frac{\left(\left(y_t - \mu_1\right) - \phi\left(y_{t-1} - \mu_2\right)\right)}{\sigma^2}.
\end{aligned}$$

Untuk memperoleh nilai $\hat{\mu}_1$ yang memaksimum fungsi *log-likelihood* maka turunan pertama dari $\mathrm{L}(\theta)$ harus sama dengan nol.

$$\frac{\partial \mathcal{L}(\theta)}{\partial \mu_1} = \sum_{t=1}^{T} \frac{1}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)} \cdot \frac{\partial f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}{\partial \mu_1} = 0$$

$$\begin{aligned}
\sum_{t=1}^{T} \Bigg[& \frac{1}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)} P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right) \frac{\left(\left(y_t - \mu_1\right) - \phi\left(y_{t-1} - \mu_1\right)\right)(1-\phi)}{\sigma^2} \\
& - \frac{1}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)} P\left(X_t = 2 \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = 2, \mathrm{Y}_{t-1};\theta\right) \frac{\left(\left(y_t - \mu_2\right) - \phi\left(y_{t-1} - \mu_1\right)\right)(\phi)}{\sigma^2} \\
& + \frac{1}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)} P\left(X_t = 3 \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = 3, \mathrm{Y}_{t-1};\theta\right) \frac{\left(\left(y_t - \mu_1\right) - \phi\left(y_{t-1} - \mu_2\right)\right)}{\sigma^2} \Bigg] = 0
\end{aligned}$$

Berdasarkan (3.1) maka dapat dituliskan

$$\sum_{t=1}^{T}\left[B(1-\phi)(y_t-\phi y_{t-1}) \; -C\phi(y_t-\mu_2-\phi y_{t-1})+D(y_t-\phi y_{t-1}+\phi\mu_2)\right.$$

$$=\mu_1\sum_{t=1}^{T}B\left[(1-\phi)-B(1-\phi)\phi+C\phi^2+D\right].$$

Sehingga diperoleh:

$$\hat{\mu}_1=\frac{\sum_{t=1}^{T}\left[B\left(1-\hat{\phi}\right)\left(y_t-\hat{\phi}y_{t-1}\right) \; -C\hat{\phi}\left(y_t-\hat{\mu}_2-\hat{\phi}y_{t-1}\right)+D\left(y_t-\hat{\phi}y_{t-1}+\hat{\phi}\hat{\mu}_2\right)\right]}{\sum_{t=1}^{T}\left[B\left(1-\hat{\phi}\right)-B\left(1-\hat{\phi}\right)\hat{\phi}+C\hat{\phi}^2+D\right]} \tag{3.2}$$

Dengan cara yang serupa diperoleh:

$$\hat{\mu}_2=\frac{\sum_{t=1}^{T}\left[C\left(y_t-\hat{\phi}y_{t-1}+\hat{\phi}\hat{\mu}_1\right)-D\hat{\phi}\left(y_t-\hat{\mu}_1-\hat{\phi}y_{t-1}\right) \; +E\left(1-\hat{\phi}\right)\left(y_t-\hat{\phi}y_{t-1}\right)\right]}{\sum_{t=1}^{T}\left[C+D\hat{\phi}^2+E\left(1-\hat{\phi}\right)-E\left(\hat{\phi}-\hat{\phi}^2\right)\right]} \tag{3.3}$$

$$\frac{\partial f\left(y_t|\Upsilon_{t-1};\theta\right)}{\partial\phi}$$

$$=P\left(X_t=1|\Upsilon_{t-1};\theta\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left((y_t-\mu_1)-\phi(y_{t-1}-\mu_1)\right)^2}{2\sigma^2}\right\}\frac{(-2)\left((y_t-\mu_1)-\phi(y_{t-1}-\mu_1)\right)\left(-(y_{t-1}-\mu_1)\right)}{2\sigma^2}$$

$$+P\left(X_t=2|\Upsilon_{t-1};\theta\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left((y_t-\mu_2)-\phi(y_{t-1}-\mu_1)\right)^2}{2\sigma^2}\right\}\frac{(-2)\left((y_t-\mu_2)-\phi(y_{t-1}-\mu_1)\right)\left(-(y_{t-1}-\mu_1)\right)}{2\sigma^2}$$

$$+P\left(X_t=3|\Upsilon_{t-1};\theta\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left((y_t-\mu_1)-\phi(y_{t-1}-\mu_2)\right)^2}{2\sigma^2}\right\}\frac{(-2)\left((y_t-\mu_1)-\phi(y_{t-1}-\mu_2)\right)\left(-(y_{t-1}-\mu_2)\right)}{2\sigma^2}$$

$$+P\left(X_t=4|\Upsilon_{t-1};\theta\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left((y_t-\mu_2)-\phi(y_{t-1}-\mu_2)\right)^2}{2\sigma^2}\right\}\frac{(-2)\left((y_t-\mu_2)-\phi(y_{t-1}-\mu_2)\right)\left(-(y_{t-1}-\mu_2)\right)}{2\sigma^2}$$

$$=P\left(X_t=1|\Upsilon_{t-1};\theta\right)f\left(y_t|X_t=1,\Upsilon_{t-1};\theta\right)\frac{\left((y_t-\mu_1)-\phi(y_{t-1}-\mu_1)\right)(y_{t-1}-\mu_1)}{\sigma^2}$$

$$+P\left(X_t=2|\Upsilon_{t-1};\theta\right)f\left(y_t|X_t=2,\Upsilon_{t-1};\theta\right)\frac{\left((y_t-\mu_2)-\phi(y_{t-1}-\mu_1)\right)(y_{t-1}-\mu_1)}{\sigma^2}$$

$$+P\left(X_t=3|\Upsilon_{t-1};\theta\right)f\left(y_t|X_t=3,\Upsilon_{t-1};\theta\right)\frac{\left((y_t-\mu_1)-\phi(y_{t-1}-\mu_2)\right)(y_{t-1}-\mu_2)}{\sigma^2}$$

$$+P\left(X_t=4|\Upsilon_{t-1};\theta\right)f\left(y_t|X_t=4,\Upsilon_{t-1};\theta\right)\frac{\left((y_t-\mu_2)-\phi(y_{t-1}-\mu_2)\right)(y_{t-1}-\mu_2)}{\sigma^2}.$$

Untuk memperoleh $\hat{\phi}$ yang memaksimum fungsi *log-likelihood* maka turunan pertama dari $L(\theta)$ harus sama dengan nol.

$$\frac{\partial \mathcal{L}(\theta)}{\partial \phi} = \sum_{t=1}^{T} \frac{1}{f\left(y_t | \mathrm{Y}_{t-1};\theta\right)} \cdot \frac{\partial f\left(y_t | \mathrm{Y}_{t-1};\theta\right)}{\partial \phi} = 0$$

$$\begin{aligned}
&\sum_{t=1}^{T}\Bigg[\frac{1}{f\left(y_t|\mathrm{Y}_{t-1};\theta\right)}P\left(X_t=1|\mathrm{Y}_{t-1};\theta\right)f\left(y_t|X_t=1,\mathrm{Y}_{t-1};\theta\right)\frac{\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_1\right)\right)\left(y_{t-1}-\mu_1\right)}{\sigma^2}\\
&\quad+\frac{1}{f\left(y_t|\mathrm{Y}_{t-1};\theta\right)}P\left(X_t=2|\mathrm{Y}_{t-1};\theta\right)f\left(y_t|X_t=2,\mathrm{Y}_{t-1};\theta\right)\frac{\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_1\right)\right)\left(y_{t-1}-\mu_1\right)}{\sigma^2}\\
&\quad+\frac{1}{f\left(y_t|\mathrm{Y}_{t-1};\theta\right)}P\left(X_t=3|\mathrm{Y}_{t-1};\theta\right)f\left(y_t|X_t=3,\mathrm{Y}_{t-1};\theta\right)\frac{\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_2\right)\right)\left(y_{t-1}-\mu_2\right)}{\sigma^2}\\
&\quad+\frac{1}{f\left(y_t|\mathrm{Y}_{t-1};\theta\right)}P\left(X_t=4|\mathrm{Y}_{t-1};\theta\right)f\left(y_t|X_t=4,\mathrm{Y}_{t-1};\theta\right)\frac{\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_2\right)\right)\left(y_{t-1}-\mu_2\right)}{\sigma^2}\Bigg]\\
&=0
\end{aligned}$$

Dari (3.1) dapat dituliskan

$$\begin{aligned}
&\sum_{t=1}^{T}\left[B(y_t-\mu_1)(y_{t-1}-\mu_1)+C(y_t-\mu_2)(y_{t-1}-\mu_1)+D(y_t-\mu_1)(y_{t-1}-\mu_2)+E(y_t-\mu_2)(y_{t-1}-\mu_2)\right]\\
&\quad=\phi\sum_{t=1}^{T}\left[B(y_{t-1}-\mu_1)^2+C(y_{t-1}-\mu_1)^2+D(y_{t-1}-\mu_2)^2+E(y_{t-1}-\mu_2)^2\right].
\end{aligned}$$

Sehingga diperoleh

$$\hat{\phi}=\frac{\sum_{t=1}^{T}\left[\left(y_t-\hat{\mu}_1\right)\left(B\left(y_{t-1}-\hat{\mu}_1\right)+D\left(y_{t-1}-\hat{\mu}_2\right)\right)+\left(y_t-\hat{\mu}_2\right)\left(C\left(y_{t-1}-\hat{\mu}_1\right)+E\left(y_t-\hat{\mu}_2\right)\right)\right]}{\sum_{t=1}^{T}\left[\left(y_{t-1}-\hat{\mu}_1\right)^2\left(B+C\right)+\left(y_{t-1}-\hat{\mu}_2\right)^2\left(D+E\right)\right]} \tag{3.4}$$

$$\begin{aligned}
&\frac{\partial f\left(y_t|\mathrm{Y}_{t-1};\theta\right)}{\partial\sigma^2}\\
&=P\left(X_t=1|\mathrm{Y}_{t-1};\theta\right)\left(-\frac{2\pi}{2}\right)\left(2\pi\sigma^2\right)^{-\frac{3}{2}}\exp\left\{\frac{-\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2}{2\sigma^2}\right\}\\
&\quad+P\left(X_t=1|\mathrm{Y}_{t-1};\theta\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2}{2\sigma^2}\right\}\frac{1\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2}{2\sigma^4}\\
&\quad+P\left(X_t=2|\mathrm{Y}_{t-1};\theta\right)\left(-\frac{2\pi}{2}\right)\left(2\pi\sigma^2\right)^{-\frac{3}{2}}\exp\left\{\frac{-\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2}{2\sigma^2}\right\}\\
&\quad+P\left(X_t=2|\mathrm{Y}_{t-1};\theta\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2}{2\sigma^2}\right\}\frac{1\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2}{2\sigma^4}\\
&\quad+P\left(X_t=3|\mathrm{Y}_{t-1};\theta\right)\left(-\frac{2\pi}{2}\right)\left(2\pi\sigma^2\right)^{-\frac{3}{2}}\exp\left\{\frac{-\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2}{2\sigma^2}\right\}\\
&\quad+P\left(X_t=3|\mathrm{Y}_{t-1};\theta\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2}{2\sigma^2}\right\}\frac{1\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2}{2\sigma^4}\\
&\quad+P\left(X_t=4|\mathrm{Y}_{t-1};\theta\right)\left(-\frac{2\pi}{2}\right)\left(2\pi\sigma^2\right)^{-\frac{3}{2}}\exp\left\{\frac{-\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2}{2\sigma^2}\right\}\\
&\quad+P\left(X_t=4|\mathrm{Y}_{t-1};\theta\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2}{2\sigma^2}\right\}\frac{1\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2}{2\sigma^4}
\end{aligned}$$

Untuk memperoleh nilai $\hat{\sigma}^2$ yang memaksimum fungsi *log-likelihood* maka turunan pertama dari $\mathrm{L}(\theta)$ harus sama dengan nol.

$$\frac{\partial L(\theta)}{\partial \sigma^2} = \sum_{t=1}^{T} \frac{1}{f\left(y_t | \mathrm{Y}_{t-1};\theta\right)} \cdot \frac{\partial f\left(y_t | \mathrm{Y}_{t-1};\theta\right)}{\partial \sigma^2} = 0$$

$$\begin{aligned}
&\sum_{t=1}^{T}\Bigg[\frac{1}{f\left(y_t|\mathrm{Y}_{t-1};\theta\right)} P\left(X_t=1|\mathrm{Y}_{t-1};\theta\right) f\left(y_t|X_t=1,\mathrm{Y}_{t-1};\theta\right)\left(\frac{-\sigma^2+\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2}{2\sigma^4}\right) \\
&\quad + \frac{1}{f\left(y_t|\mathrm{Y}_{t-1};\theta\right)} P\left(X_t=2|\mathrm{Y}_{t-1};\theta\right) f\left(y_t|X_t=2,\mathrm{Y}_{t-1};\theta\right)\left(\frac{-\sigma^2+\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2}{2\sigma^4}\right) \\
&\quad + \frac{1}{f\left(y_t|\mathrm{Y}_{t-1};\theta\right)} P\left(X_t=3|\mathrm{Y}_{t-1};\theta\right) f\left(y_t|X_t=3,\mathrm{Y}_{t-1};\theta\right)\left(\frac{-\sigma^2+\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2}{2\sigma^4}\right) \\
&\quad + \frac{1}{f\left(y_t|\mathrm{Y}_{t-1};\theta\right)} P\left(X_t=4|\mathrm{Y}_{t-1};\theta\right) f\left(y_t|X_t=4,\mathrm{Y}_{t-1};\theta\right)\left(\frac{-\sigma^2+\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2}{2\sigma^4}\right)\Bigg] \\
&= 0
\end{aligned}$$

Berdasarkan (3.1) dapat dituliskan

$$\begin{aligned}
&\sum_{t=1}^{T}\Big[B\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2 + C\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2 + D\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2 \\
&\quad + E\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2\Big] \\
&= \sigma^2 \sum_{t=1}^{T}\left[B+C+D+E\right]\cdot
\end{aligned}$$

Sehingga diperoleh

$$\hat{\sigma}^2 = \frac{\sum_{t=1}^{T}\left[B\left(\left(y_t-\hat{\mu}_1\right)-\hat{\phi}\left(y_{t-1}-\hat{\mu}_1\right)\right)^2 + C\left(\left(y_t-\hat{\mu}_2\right)-\hat{\phi}\left(y_{t-1}-\hat{\mu}_1\right)\right)^2 + D\left(\left(y_t-\hat{\mu}_1\right)-\hat{\phi}\left(y_{t-1}-\hat{\mu}_2\right)\right)^2 + E\left(\left(y_t-\hat{\mu}_2\right)-\hat{\phi}\left(y_{t-1}-\hat{\mu}_2\right)\right)^2\right]}{\sum_{t=1}^{T}\left[B+C+D+E\right]} \qquad (3.5)$$

Untuk menduga parameter *P* digunakan formula (3.6) berikut yang diperoleh dari Hamilton (1990).

$$\hat{p}_{ji} = \frac{\sum_{t=2}^{T} P\left(X_t = j, X_{t-1} = i \mid \mathrm{Y}_T;\hat{\theta}\right)}{\sum_{t=2}^{T} P\left(X_{t-1} = i \mid \mathrm{Y}_T;\hat{\theta}\right)} \qquad (3.6)$$

di mana menurut Kim (1994)

$$
\begin{aligned}
&P\left(X_t = j, X_{t-1} = i \mid \mathrm{Y}_T; \hat{\theta}\right) \\
&= P\left(X_t = j \mid \mathrm{Y}_T; \hat{\theta}\right) \times P\left(X_{t\text{-}1} = i \mid X_t = j, \mathrm{Y}_T; \hat{\theta}\right) \\
&\approx P\left(X_t = j \mid \mathrm{Y}_T; \hat{\theta}\right) \times P\left(X_{t\text{-}1} = i \mid X_t = j, \mathrm{Y}_T; \hat{\theta}\right) \\
&= \frac{P\left(X_t = j \mid \mathrm{Y}_T; \hat{\theta}\right) \times P\left(X_{t\text{-}1} = i, X_t = j \mid \mathrm{Y}_T; \hat{\theta}\right)}{P\left(X_t = j \mid \mathrm{Y}_T; \hat{\theta}\right)} \\
&= \frac{P\left(X_t = j \mid \mathrm{Y}_T; \hat{\theta}\right) \times P\left(X_{t\text{-}1} = i \mid \mathrm{Y}_T; \hat{\theta}\right) \times P\left(X_t = j \mid X_{t\text{-}1} = i\right)}{P\left(X_t = j \mid \mathrm{Y}_T; \hat{\theta}\right)} \\
&= \frac{\hat{\varsigma}_{t|T}^{(j)} \cdot \hat{\varsigma}_{t-1|T}^{(i)} \cdot a_{ji}}{\hat{\varsigma}_{t|T}^{(j)}}.
\end{aligned}
\tag{3.7}
$$

dan

$$
P\left(X_{t-1} = i \mid \mathrm{Y}_T; \hat{\theta}\right) = \sum_{j=1}^{4} P\left(X_t = j, X_{t-1} = i \mid \mathrm{Y}_T; \hat{\theta}\right) = \sum_{j=1}^{4} \frac{\hat{\varsigma}_{t|T}^{(j)} \cdot \hat{\varsigma}_{t-1|T}^{(i)} \cdot a_{ji}}{\hat{\varsigma}_{t|T}^{(j)}}.
$$

4. PENDUGAAN ULANG PARAMETER DAN ALGORITME

Karena persamaan (3.1) sampai (3.7) tak linear, maka tidak mungkin untuk memperoleh $\hat{\theta}$ secara analitik sebagai fungsi dari $\{Y_1, Y_2, Y_3, \cdots, Y_T\}$. Sehingga untuk mencari penduga kemungkinan maksimum digunakan algoritme iteratif dari metode EM.

4.1 Metode *Expectation Maximization* (Metode EM)

Algoritme EM dikembangkan oleh Baum and Petrie (1966) dengan ide dasar sebagai berikut.

Misalkan $\{P_\theta : \theta \in \Theta\}$ adalah koleksi ukuran peluang yang terdefinisi pada ruang (Ω, G) dan kontinu absolut terhadap P_0. Misalkan $\mathrm{Y} \subset \mathrm{G}$. Definisikan fungsi *likelihood* untuk menentukan penduga parameter θ berdasarkan informasi Y sebagai

$$
L(\theta) = E_0\left[\left.\frac{dP_\theta}{dP_0}\right| \mathrm{Y}\right]
$$

dan penduga maksimum *likelihood* didefinisikan sebagai

$$
\hat{\theta} \in \arg\max_{\theta \in \Theta} L(\theta).
$$

Secara umum penduga maksimum likelihood $\hat{\theta}$ sulit dihitung secara langsung. Algoritme EM memberikan suatu metode iteratif untuk mengaproksimasi $\hat{\theta}$, dengan prosedur sebagai berikut.

Langkah 1: Set $p=0$ dan pilih $\hat{\theta}_0$.

Langkah 2: [Langkah-E]

Set $\theta^* = \hat{\theta}_p$ dan hitung $Q\left(\theta,\theta^*\right) = E_{\theta^*}\left[\log \frac{dP_\theta}{dP_{\theta^*}} \middle| \mathrm{Y}\right]$.

Langkah 3: [Langkah-M]

Tentukan $\hat{\theta}_{p+1} \in \arg\max_{\theta\in\Theta} Q\left(\theta,\theta^*\right)$.

Langkah 4: $p \leftarrow p+1$

Ulangi langkah 2 sampai kriteria berhenti dipenuhi.

Catatan:

1. Barisan $\left\{\hat{\theta}_p : p \geq 0\right\}$ memberikan barisan $\left\{L\left(\hat{\theta}_p\right): p \geq 0\right\}$ yang tak turun.
2. Menurut ketaksamaan Jensen,
$$Q\left(\hat{\theta}_{p+1},\hat{\theta}_p\right) \leq \log L(\hat{\theta}_{p+1}) - \log L(\hat{\theta}_p).$$
3. $Q\left(\theta,\theta^*\right)$ disebut *pseudo-loglikelihood* bersyarat.

4.2 Pendugaan ulang parameter menggunakan metode EM

Dengan menggunakan metode EM diperoleh algoritme untuk menduga ulang parameter model. Algoritme tersebut sebagai berikut.

Langkah 1:

Tentukan banyaknya data (T) yang akan diamati serta tentukan juga nilai $\left(y_0, y_1, y_2, \cdots, y_T\right)$ dan matriks transisi

$$\mathbf{P} = \begin{bmatrix} p_{11}^* & 0 & p_{11}^* & 0 \\ p_{21}^* & 0 & p_{21}^* & 0 \\ 0 & p_{12}^* & 0 & p_{12}^* \\ 0 & p_{22}^* & 0 & p_{22}^* \end{bmatrix}.$$

Beri nilai awal bagi $\hat{\theta}$ yang dilambangkan dengan $\hat{\theta}^{(m)} = \left(\hat{\mu}_1, \hat{\mu}_2, \hat{\phi}, \hat{\sigma}^2\right)$.

Langkah 2:

Cari fungsi kerapatan bersyarat bagi y_T untuk setiap $t = 1, 2, \cdots, T$ dengan cara

$$\eta_t = \begin{bmatrix} f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right) \\ f\left(y_t \middle| X_t = 2, \mathrm{Y}_{t-1};\theta\right) \\ f\left(y_t \middle| X_t = 3, \mathrm{Y}_{t-1};\theta\right) \\ f\left(y_t \middle| X_t = 4, \mathrm{Y}_{t-1};\theta\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2}{2\sigma^2}\right\} \\ \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_1\right)\right)^2}{2\sigma^2}\right\} \\ \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left(\left(y_t-\mu_1\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2}{2\sigma^2}\right\} \\ \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-\left(\left(y_t-\mu_2\right)-\phi\left(y_{t-1}-\mu_2\right)\right)^2}{2\sigma^2}\right\} \end{bmatrix}$$

Langkah 3:

Penarikan kesimpulan optimal dan peramalan untuk setiap waktu *t* pada contoh dapat diperoleh melalui iterasi:

3.1. Tentukan nilai awal bagi $\hat{\xi}_{t|t-1}$ yang dilambangkan dengan $\hat{\xi}_{1|0}$

$$\hat{\xi}_{1|0} = \begin{bmatrix} \dfrac{-p_{12}^{*}p_{11}^{*}}{p_{11}^{*}-p_{12}^{*}-1} \\ \dfrac{p_{11}^{*}-1-p_{11}^{*}p_{22}^{*}+p_{22}^{*}}{p_{11}^{*}-p_{12}^{*}-1} \\ \dfrac{p_{12}^{*}\left(p_{11}^{*}-1\right)}{p_{11}^{*}-p_{12}^{*}-1} \\ \dfrac{\left(p_{11}^{*}-1\right)p_{22}^{*}}{p_{11}^{*}-p_{12}^{*}-1} \end{bmatrix}$$

3.2. Beri nilai awal $i=1$

3.3. Untuk $t=i$, cari nilai dari

$$f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}^{(m)}\right)=\mathbf{1}'\left(\hat{\xi}_{t|t-1}\right)\otimes\eta_t$$

$$\hat{\xi}_{t|t}=\begin{bmatrix} P\left(X_t=1\middle|\mathrm{Y}_{t-1};\hat{\theta}\right) \\ P\left(X_t=2\middle|\mathrm{Y}_{t-1};\hat{\theta}\right) \\ P\left(X_t=3\middle|\mathrm{Y}_{t-1};\hat{\theta}\right) \\ P\left(X_t=4\middle|\mathrm{Y}_{t-1};\hat{\theta}\right) \end{bmatrix}=\frac{\left(\hat{\xi}_{t|t-1}\otimes\eta_t\right)}{\mathbf{1}'\left(\hat{\xi}_{t|t-1}\otimes\eta_t\right)}$$

$$\hat{\xi}_{t+1|t}=\begin{bmatrix} P\left(X_{t+1}=1\middle|\mathrm{Y}_t;\hat{\theta}\right) \\ P\left(X_{t+1}=2\middle|\mathrm{Y}_t;\hat{\theta}\right) \\ P\left(X_{t+1}=3\middle|\mathrm{Y}_t;\hat{\theta}\right) \\ P\left(X_{t+1}=4\middle|\mathrm{Y}_t;\hat{\theta}\right) \end{bmatrix}=\mathbf{P}\cdot\hat{\xi}_{t|t}$$

$i=i+1$

3.4. Ulangi mulai dari langkah (3.3)

Stop jika $t=T$.

Lanjutkan ke langkah 4.

Langkah 4:

Misalkan

$$B = \frac{1}{f\left(y_t \middle| Y_{t-1};\hat{\theta}\right)} P\left(X_t = 1 \middle| Y_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 1, Y_{t-1};\hat{\theta}\right)$$

$$C = \frac{1}{f\left(y_t \middle| Y_{t-1};\hat{\theta}\right)} P\left(X_t = 2 \middle| Y_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 2, Y_{t-1};\hat{\theta}\right)$$

$$D = \frac{1}{f\left(y_t \middle| Y_{t-1};\hat{\theta}\right)} P\left(X_t = 3 \middle| Y_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 3, Y_{t-1};\hat{\theta}\right)$$

$$E = \frac{1}{f\left(y_t \middle| Y_{t-1};\hat{\theta}\right)} P\left(X_t = 4 \middle| Y_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 4, Y_{t-1};\hat{\theta}\right)$$

Cari nilai dari:

$$\hat{\mu}_1 = \frac{\sum_{t=1}^{T}\left[B\left(1-\hat{\phi}\right)\left(y_t - \hat{\phi} y_{t-1}\right) - C\hat{\phi}\left(y_t - \hat{\mu}_2 - \hat{\phi} y_{t-1}\right) + D\left(y_t - \hat{\phi} y_{t-1} + \phi\hat{\mu}_2\right)\right]}{\sum_{t=1}^{T}\left[B\left(1-\hat{\phi}\right) - B\left(1-\hat{\phi}\right)\hat{\phi} + C\hat{\phi}^2 + D\right]}$$

$$\hat{\mu}_2 = \frac{\sum_{t=1}^{T}\left[C\left(y_t - \hat{\phi} y_{t-1} + \hat{\phi}\hat{\mu}_1\right) - D\hat{\phi}\left(y_t - \hat{\mu}_1 - \hat{\phi} y_{t-1}\right) + E\left(1-\hat{\phi}\right)\left(y_t - \hat{\phi} y_{t-1}\right)\right]}{\sum_{t=1}^{T}\left[C + D\hat{\phi}^2 + E\left(1-\hat{\phi}\right) - E\left(\hat{\phi} - \hat{\phi}^2\right)\right]}$$

$$\hat{\phi} = \frac{\sum_{t=1}^{T}\left[\left(y_t - \hat{\mu}_1\right)\left(B\left(y_{t-1} - \hat{\mu}_1\right) + D\left(y_{t-1} - \hat{\mu}_2\right)\right) + \left(y_t - \hat{\mu}_2\right)\left(C\left(y_{t-1} - \hat{\mu}_1\right) + E\left(y_t - \hat{\mu}_2\right)\right)\right]}{\sum_{t=1}^{T}\left[\left(y_{t-1} - \hat{\mu}_1\right)^2\left(B + C\right) + \left(y_{t-1} - \hat{\mu}_2\right)^2\left(D + E\right)\right]}$$

$$\hat{\sigma}^2 = \frac{\sum_{t=1}^{T}\left[B\left(\left(y_t - \hat{\mu}_1\right) - \hat{\phi}\left(y_{t-1} - \hat{\mu}_1\right)\right)^2 + C\left(\left(y_t - \hat{\mu}_2\right) - \hat{\phi}\left(y_{t-1} - \hat{\mu}_1\right)\right)^2 + D\left(\left(y_t - \hat{\mu}_1\right) - \hat{\phi}\left(y_{t-1} - \hat{\mu}_2\right)\right)^2 + E\left(\left(y_t - \hat{\mu}_2\right) - \hat{\phi}\left(y_{t-1} - \hat{\mu}_2\right)\right)^2\right]}{\sum_{t=1}^{T}\left[B + C + D + E\right]}$$

Langkah 5:

Beri nama parameter yang dihasilkan pada langkah 4 dengan $\hat{\theta}^{(m+1)} = \left(\hat{c}_1, \hat{c}_2, \hat{\phi}_1, \hat{\phi}_2, \hat{\sigma}^2\right)$

Langkah 6:

Cari $P = (P_{ij})$ yang baru menggunakan hasil Kim, C.J (1994) dan Hamilton, J. D. (1994), yaitu:

$$\hat{\xi}^{(j)}_{t|T} = \hat{\xi}^{(j)}_{t|t} \odot \left\{ \mathbf{P}' \cdot \left[\hat{\xi}^{(j)}_{t+1|T} (\div) \hat{\xi}^{(j)}_{t+1|t} \right] \right\}$$

$$P\left(X_t = j, X_{t-1} = i \mid Y_T\right) \approx \frac{\hat{\xi}^{(j)}{}_{t|T} \times \hat{\xi}^{(i)}{}_{t-1|t-1} \times p_{ij}}{\hat{\xi}^{(j)}{}_{t|t}}$$

$$P\left(X_{t-1} = i \mid Y_t;\hat{\theta}\right) = \sum_{j=1}^{2} P\left(X_t = j, X_{t-1} = i \mid Y_T\right)$$

$$\hat{p}_{ji} = \frac{\sum_{t=2}^{T} P\left(X_t = j, X_{t-1} = i \mid Y_T;\hat{\theta}\right)}{\sum_{t=2}^{T} P\left(X_{t-1} = i \mid Y_T;\hat{\theta}\right)} = \frac{\sum_{t=2}^{T} \frac{\hat{\xi}^{(j)}_{t|T} \times \hat{\xi}^{(i)}_{t-1|t-1} \times p_{ij}}{\hat{\xi}^{(j)}_{t|t}}}{\sum_{t=2}^{T}\sum_{j=1}^{2} \frac{\hat{\xi}^{(j)}_{t|T} \times \hat{\xi}^{(i)}_{t-1|t-1} \times p_{ij}}{\hat{\xi}^{(j)}_{t|t}}}$$

Langkah 7:

Selama $k < T$, ulangi mulai dari langkah 2. Gunakan parameter yang sudah dihasilkan untuk mencari nilai harapan

$$E\left[y_{t+1} \mid X_{t+1} = j, X_t = i, Y_t; \theta\right] = \mu_j + \phi y_t - \mu_i \text{ .}$$

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