PERFORMANCE LIMITATION OF SYSTEMS UNDER SIGNAL-TO-NOISE CONSTRAINED CHANNELS

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Abstract. This paper re-discusses [1] and [11], where the problems of feedback stabilization over a signal-to-noise ratio (SNR) constrained channel are studied. The first paper considers both continuous and discrete-time minimum phase systems, while the second extends the results to non-minimum phase ones and proposes a linear time-varying feedback strategies to eliminate the effect of non-minimum phase zeros in SNR limited stabilization. In general, the limitations on the ability to stabilize a plant over an SNR constrained channel are imposed mainly by unstable poles and non-minimum phase zeros of the plant.
Keywords: Communication constraints, SNR constrained channel, performance limitations, unstable/non-minimum phase systems.

1. Introduction

In the recent years, there is a growing attention related to the research activity in feedback control with communication constraints. A number of studies have considered in the problem of feedback stabilization subject to some constraints including quantization effects [2, 9], limited information [5, 6, 10, 12], bandwidth constraints [4], variable time delay, and missing data. Particularly, a new control design methodology, which relies on the possibility of changing the sensitivity of the quantizer, is proposed in [2] for feedback stabilization problems of linear time-invariant control systems with saturating quantized measurements. While that with limited information is considered in [10] by using sampled encoded measurements of the state or output and in [6] by solving a special linear quadratic regulator (LQR) problem.

The main motivation for the work in this paper is the observation that a bit rate limitation may be due to SNR channel limitations, as quantified by Shannon's theorem on the capacity of a channel. This paper offers an alternate view point based on a SNR limitation in the feedback additive white Gaussian noise channel. By reducing the considered communication link into the noisy channel, the fundamental

Figure 1. A communication link.

Figure 2. A noisy channel.

limitations arising from a simple ideal channel model are quantified. An interesting link between this SNR result and a related bit rate limited result in [12] is noted. It is also demonstrated that the limitations on the required SNR depends on the unstable poles and non-minimum phase zeros of the plant. Then a linear time-varying feedback strategy is implemented to eliminate the effect contributed by non-minimum phase zeros.

The rest of the paper is organized as follows. Some preliminaries are given in Section 2. The main results for minimum and non-minimum phase systems are devoted in Section 3 and 4, respectively. The implementation of linear time-varying feedback strategy is described in Section 5. Concluding statements are in Section 6.

2. Preliminaries

In this paper, a feedback control system in which there exists a communication link is considered, see Fig. 1. In digital communication, the link consists of some pre and post processing equipments for the signal that are sent through the communication channel, which might be in the form of filter, A-D converter, coder, modulator, decoder, demodulator, and D-A converter.

In this study, an SNR constrained channel will be considered and all pre and post signal processing are restricted to LTI filtering and D-A/A-D type operations. Thus, the communication link simplifies to the noisy channel itself, Fig. 2. Here, n is a zero mean additive white Gaussian noise with intensity Ω, i.e.,

$$\mathcal{E}[n(t)] = 0, \;\; \mathcal{E}[n'(t)n(k)] = \Omega\delta(t-k),$$

where $\mathcal{E}[\cdot]$ represents the expectation operator and δ is the unitary impulse function.

Let a discrete-time LTI system is given by

$$\begin{aligned} x(k+1) &= A_d x(k) + B_d u(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

where x, u, y are state, input, and output variables, respectively, and A_d, B_d, C are their corresponding matrices. Then a necessary and sufficient condition for the asymptotic feedback stabilizability of (1) through a digital channel of limited bit rate capacity with the data rate R (bits per interval) is given in [12] by

$$R \geq \sum_{|\lambda_k|\geq 1} \log_2 |\lambda_k|, \tag{2}$$

FIGURE 3. Control system with communication link.

FIGURE 4. Simplified control system with communication link.

where λ_k are the unstable eigenvalues of A_d (or, the unstable poles of the corresponding plant, P). Suppose that (1) arises from the discretization process with sampling time T over a continuous-time system with unstable eigenvalues $p_k, k = 1, \ldots, N_p$, then the continuous-time counterpart of the bound (2) is

$$\frac{R}{T} \geq \log_2 e \sum_{\mathrm{Re}(p_k) \geq 0} \mathrm{Re}(p_k). \tag{3}$$

3. Minimum Phase Systems

To study the feedback stabilization of minimum phase plants a standard feedback control setup in Fig. 3 is considered, in which there exists a communication link between compensator and actuator depicted by Fig. 1.

Fig. 3 then can be simplified by Fig. 4, where P represents the plant, K the compensator, u the control input, y the system output, and n the noise signal. It is assumed that the control input signal u is a stationary stochastic process with root mean square (RMS) value

$$\|u\|_{RMS} = \sqrt{\mathcal{E}[u'u]}. \tag{4}$$

The energy of the signal u is defined as $\|u\|_{RMS}^2$ and is assumed to satisfy the constraint

$$\|u\|_{RMS}^2 < \mathcal{U} \tag{5}$$

for some predetermined value $\mathcal{U} > 0$. Such an energy constraint may arise either from electronic hardware limitations or regulatory constraints introduced to minimize interference to other communication system users.

We restrict attention to the single-input single-output (SISO) case where the compensator, pre and post compensators are linear time-invariant and free to the designer. We assume that the feedback loop is internally stable.

The output feedback stabilization problem is then can be stated as a problem of finding the smallest value of $\|u\|_{RMS}$ with respect to the class of all stabilizing compensators in $\mathcal{K}$. It is well-known that $\mathcal{K}$ is characterized by Youla parameterization.

It is possible to verify that

$$\|u\|_{RMS}^2 = \|\mathcal{T}\|_2^2 \, \Omega, \tag{6}$$

where $\mathcal{T}$ is the closed-loop transfer function between n and y, i.e.,

$$\mathcal{T} = \frac{PK}{1+PK}. \tag{7}$$

Remark 3.1. *According to Fig. 4 and (6), the problem addressed in this section in many senses can be viewed as an energy regulation problem, which is previously discussed in* [8] *for continuous-time systems, and in* [7] *for discrete-time ones.*

Now we are ready to present our results. First for continuous-time case and then for discrete-time case.

Theorem 3.1. ***(Continuous-time)*** *Suppose that the plant $P(s)$ is minimum phase and has unstable poles $p_k, k = 1, \ldots, N_p$. Then*

$$\inf_{K\in\mathcal{K}} \|\mathcal{T}(s)\|_2^2 = 2\sum_{k=1}^{N_p} Re(p_k). \tag{8}$$

Then according to (5) and (6), the SNR channel must satisfy

$$\frac{\mathcal{U}}{2\Omega} \geq \sum_{k=1}^{N_p} \mathrm{Re}(p_k).$$

Meanwhile, the capacity $\mathcal{C}$ of the channel as in Fig. 2, with infinite bandwidth, energy constraint (5), and noise spectral density Ω can be made arbitrarily close to

$$\mathcal{C} = \frac{\mathcal{U}\log_2 e}{2\Omega} \tag{9}$$

bits per second. Thus, the maximum channel capacity (9) permitted by Shannon's theorem[1] must satisfy

$$\mathcal{C} \geq \log_2 e \sum_{k=1}^{N_p} \mathrm{Re}(p_k),$$

which gives the same bound in (3).

Theorem 3.2. ***(Discrete-time)*** *Suppose that the plant $P(z)$ is minimum phase (possibly strictly proper with relative degree* 1*) and has unstable poles $\lambda_k, k = 1, \ldots, N_\lambda$. Then*

$$\inf_{K\in\mathcal{K}} \|\mathcal{T}(s)\|_2^2 = \prod_{k=1}^{N_\lambda} |\lambda_k|^2 - 1. \tag{10}$$

[1]The Shannon theorem states that given a noisy channel with channel capacity C and information transmitted at a rate R, then if $R < C$ there exist codes that allow the probability of error at the receiver to be made arbitrarily small. This means that theoretically, it is possible to transmit information nearly without error at any rate below a limiting rate, C.

FIGURE 5. Control system with communication link in the feedback loop.

FIGURE 6. Simplified control system with communication link in the feedback loop.

Theorem 3.2 shows that the SNR discrete-time channel must satisfy

$$\frac{\mathcal{U}}{\Omega} \geq \prod_{k=1}^{N_\lambda} |\lambda_k|^2 - 1.$$

And since

$$\log_2\left(1 + \frac{\mathcal{U}}{\Omega}\right) \geq \log_2 \prod_{k=1}^{N_\lambda} |\lambda_k|^2 = 2\sum_{k=1}^{N_\lambda} \log_2 |\lambda_k|,$$

the maximum channel capacity $\mathcal{C}$ permitted by Shannon's theorem obeys

$$\mathcal{C} \geq \sum_{k=1}^{N_\lambda} \log_2 |\lambda_k|,$$

where

$$\mathcal{C} = \frac{1}{2}\log_2\left(1 + \frac{\mathcal{U}}{\Omega}\right).$$

We again recover the bound (2) on the smallest data rate required for stabilization.

4. Non-minimum Phase Systems

In this section, the feedback control setup depicted by Fig. 5 and its simplified version in is Fig. 6 are considered, in which there exists a communication link or noisy channel in the feedback loop.

We extend the analysis of SNR limitations to the non-minimum phase case. We define

$$\|y\|_{RMS} = \sqrt{\mathcal{E}[y'y]}. \tag{11}$$

and assume a given energy constraint $\mathcal{Y}$, such that it is required

$$\|y\|^2_{RMS} < \mathcal{Y} \tag{12}$$

for some predetermined value $\mathcal{Y} > 0$. The stabilization problem addressed in this section is then can be stated as a problem of finding the smallest value of $\|y\|_{RMS}$. Further we may write

$$\|y\|^2_{RMS} = \|\mathcal{T}\|^2_2\, \Omega, \tag{13}$$

Remark 4.1. *Since $\|y\|^2_{RMS}$ is also completely determined by the complementary sensitivity function $\mathcal{T}$ as in the preceding case, then the problem can be similarly treated as an energy regulation problem of non-minimum phase systems* [7, 8].

Theorem 4.1. ***(Continuous-time*** [8]***)*** *Suppose that the plant $P(s)$ has unstable poles $p_k, k = 1, \ldots, N_p$ and non-minimum phase zeros $z_k, k = 1, \ldots, N_z$. Then*

$$\inf_{K\in\mathcal{K}} \|\mathcal{T}(s)\|_2^2 = 2\sum_{k=1}^{N_p} Re(p_k) + \sum_{j=1}^{N_z}\sum_{k=1}^{N_z} \frac{4Re(z_j)Re(z_k)}{\bar{a}_j a_k(\bar{z}_j + z_k)}\bar{\alpha}_j\alpha_k, \tag{14}$$

where

$$a_j = \prod_{j\neq k} \frac{z_k - z_j}{z_k + \bar{z}_j}, \ \alpha_j = 1 - \prod_{k=1}^{N_p} \frac{z_j + \bar{p}_k}{z_j - p_k}.$$

Theorem 4.2. ***(Discrete-time*** [7]***)*** *Suppose that the plant $P(z)$ has unstable poles $\lambda_k, k = 1, \ldots, N_\lambda$ and non-minimum phase zeros $\eta_k, k = 1, \ldots, N_\eta$. Then*

$$\inf_{K\in\mathcal{K}} \|\mathcal{T}(z)\|_2^2 = \left(\prod_{k=1}^{N_\lambda} |\lambda_k|^2 - 1\right) + \sum_{j=1}^{N_\eta}\sum_{k=1}^{N_\eta} \frac{(|\eta_j|^2-1)(|\eta_k|^2-1)}{\bar{b}_i b_j(\bar{\eta}_j\eta_k - 1)}\bar{\beta}_j\beta_k, \tag{15}$$

where

$$b_j = \prod_{j\neq k} \frac{\eta_j - \eta_k}{\eta_j\bar{\eta}_k - 1}, \ \beta_j = \prod_{k=1}^{N_\lambda} \bar{\lambda}_k - \prod_{k=1}^{N_\lambda} \frac{\bar{\lambda}_k\eta_j - 1}{\eta_j - \lambda_k}.$$

Theorems 4.1 and 4.2 show that the plant non-minimum phase zeros give additional restriction in the achievable $\mathcal{H}_2$ norm of $\mathcal{T}$. It gives additional SNR constraint for stabilization.

5. Linear Time Varying Feedback for Stabilization

In this section, we propose a specific type of linear time varying (LTV) feedback strategy in order to avoid the additional effect imposed by plant non-minimum phase zeros as indicated in Theorem 4.1. We consider the feedback control system with communication link depicted in Fig. 5, in which the link is illustrated by Fig. 1. The post-signal processing evolves an averager F_T, sampler S_T, and hold H_T, as depicted in Fig. 7.

- **Averager:** the averager is an LTI filter with the following operation
$$F_T(s) = \frac{1-e^{-sT}}{sT}, \quad y_f(t) = \frac{1}{T}\int_{t-T}^{t} y_r(\tau)d\tau.$$
- **Sampler:** $S_T: \ w_k = y_f(kT), \ k = 0, 1, \ldots.$

FIGURE 7. Post-signal processing for LTV feedback.

FIGURE 8. Pre-signal processing for LTV feedback.

- **Hold:** the hold is zero order hold (ZOH), where $H_T : e(t) = -w_k,\ t \in [kT, (k+1)T)$.

The pre-signal processing is more complicated since it contains a deadbeat state observer, state feedback $K(z)$, and hold H_T, as illustrated in Fig. 8.

- **State Feedback:** $v_k = K(z)\hat{x}_k$.
- **Deadbeat State Observer:** the observer takes continuous measurement output $y(t)$ and produces sampled-data estimates of the extended state

$$X(t) = \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$$

according to:

$$\begin{aligned} & z(kT^+) = 0,\ k = 0, 1, \ldots \\ & \dot{z}(t) = \mathcal{A}'z(t) + \mathcal{C}'y(t),\ t \in (kT, (k+1)T) \\ & \hat{X}_k = \begin{pmatrix} \hat{x}_k \\ \hat{u}_{k-1} \end{pmatrix} = W_T^{-1} z(kT^-), \end{aligned}$$

where z is the state observer variable and W_T is the finite time observability Grammian

$$W_T = \int_0^T e^{-\mathcal{A}'t}\mathcal{C}'\mathcal{C}e^{-\mathcal{A}t}dt$$

and

$$\mathcal{A} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix},\ \mathcal{C} = (C\ \ 0),$$

with A, B, C are the state space representation matrices of the plant P.

We now demonstrate that under suitable assumptions, the control system with LTV feedback can be simplified to a discrete-time, LTI, state feedback problem.

Assumption 5.1. ***(Non-zero steady state plant gain)*** *We assume that $P(0) \neq 0$, i.e.,*

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

is full rank.

Note that this assumption is required in the following development since for simplicity we restrict attention to the case of a zero-order hold input.

Proposition 5.1. *Consider the post and pre signal processing in Fig. 7 and 8. Then*

$$w_k = v_{k-1} + n_k,$$

where n_k is a white noise process with zero mean and variance $\mathcal{E}[n_k^2] = \Omega/T$.

Proposition 5.2. *Suppose that the compensator is an identity operator and define $u_k = -e_k$. Assuming $(A \;\; C)$ is observable, then W_T is well defined and*

$$\begin{aligned} \hat{x}_k &= x_k, \\ \hat{u}_{k-1} &= u_{k-1}. \end{aligned}$$

Note that under Proposition 5.1 and 5.2 it is revealed that the respecting system can be expressed as a discrete-time delayed system

$$\begin{aligned} x_{k+1} &= A_T x_k + B_T u_k, \\ u_k &= -K(z)x_{k-1} - n_k, \end{aligned} \tag{16}$$

where A_T and B_T are the appropriate ZOH discretization of A and B, respectively:

$$A_T = e^{AT}, \; B_T = \int_0^T e^{At} B \, dt.$$

Hence, in the case of ideal stabilization, we are able to reduce the problem to a discrete-time delayed state feedback problem, i.e., we have reduce the continuous output feedback stabilization problem to that of solving

$$\inf_{K(z)\in\mathcal{K}} \mathcal{E}[(Kx_k)^2] = \inf_{K(z)\in\mathcal{K}} \|\mathcal{T}_k(z)\|_2^2 \, \Omega/T, \tag{17}$$

where $\mathcal{T}_k(z)$ is the transfer function from n_k to $K(z)x_k$.

Theorem 5.1. *Consider the delayed system (16). Then*

$$\inf_{K\in\mathcal{K}} \|\mathcal{T}_k(z)\|_2^2 = \left(\prod_{k=1}^{N_\lambda} |\lambda_k|^2 - 1 \right) + \Psi, \tag{18}$$

where

$$\Psi = \left(\prod_{k=1}^{N_\lambda} |\lambda_k|^2 \right) \left| \sum_{k=1}^{N_\lambda} \frac{|\lambda_k|^2 - 1}{\lambda_k} \right|^2 .$$

Theorem 5.1 shows that the system of LTV feedback can be stabilized without exceeding the energy constraint (12) if and only if

$$\frac{\mathcal{Y}}{\Omega} \geq \frac{\left(\prod_{k=1}^{N_\lambda} |\lambda_k|^2 - 1 \right) + \Psi}{T}.$$

Further we can show that

$$\lim_{T\to 0}\frac{\left(\prod_{k=1}^{N_\lambda}|\lambda_k|^2-1\right)+\Psi}{T}=2\sum_{k=1}^{N_p}\mathrm{Re}(p_k),$$

which means that, if the SNR constraint satisfies

$$\frac{\mathcal{Y}}{2\Omega}\geq\sum_{k=1}^{N_p}\mathrm{Re}(p_k),$$

then there exists a sufficiently small T such that the system is stabilizable by LTV feedback strategies.

Remark 5.1. *Theorem 5.1 is derived by assuming that the discretized plant $P(z)$ has (minimum) relative degree 2 (without having any other NMP zeros). It means that $P(s)$ also has relative degree 2 (without having any other NMP zeros). In our understanding, LTV feedback strategy is implemented only to tackle effects caused by zeros at infinity. This is less useful since in continuous-time problem, zeros at infinity do not give effect as in Theorem 4.1,*

$$\sum_{j=1}^{2}\sum_{k=1}^{2}\frac{4Re(z_j)Re(z_k)}{\bar{h}_j h_k(\bar{z}_j+z_k)}\bar{\alpha}_j\alpha_k\bigg|_{z=\infty}=0.$$

Remark 5.2. *We may derive Ψ in (18) by using the second term of (15) in Theorem 4.2. For instance, if $P(z)$ has two real unstable poles λ_1 and λ_2, then*

$$\begin{aligned}\sum_{j=1}^{1}\sum_{k=1}^{1}\frac{(|\eta_j|^2-1)(|\eta_k|^2-1)}{\bar{b}_i b_j(\bar{\eta}_j\eta_k-1)}\bar{\beta}_j\beta_k\bigg|_{\eta=\infty} &= (\lambda_1\lambda_2^2+\lambda_1^2\lambda_2-\lambda_1-\lambda_2)^2\\ &= [\lambda_1(\lambda_2^2-1)+\lambda_2(\lambda_1^2-1)]^2\\ &= \lambda_1^2\lambda_2^2\left(\frac{\lambda_1^2-1}{\lambda_1}+\frac{\lambda_2^2-1}{\lambda_2}\right)^2.\end{aligned}$$

6. Conclusion

In this paper, the feedback stabilization problem over SNR constrained channel has been considered. In both continuous and discrete-time minimum phase systems there are limitations on the ability to stabilize imposed by plant unstable poles, while for non-minimum phase cases, plant non-minimum phase zeros give additional restrictions. In the ideal case, this additional effects may be essentially removed by the use of linear time varying feedback strategies. We have shown that the problem can also be seen as an energy regulation control problem.

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