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DEVELOPING PROOF-BASED LEARNING USING APOS THEORY APPROACH IN EXPONENTIAL FOR ENHANCING STUDENTS' REASONING ABILITY

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Abstrak

Pembelajaran berbasis bukti merupakan pembelajaran matematika melalui pembuktian agar memperkuat konsep siswa. Penggunaan teori APOS (Aksi, Proses, Objek, dan Skema) bertujuan untuk mendeskripsikan struktur mental siswa dirangkum dalam HLT (*Hypothetical Learning Trajectory*). Berdasarkan dari kepentingan pembelajaran matematika, siswa dituntut untuk mempunyai kemampuan penalaran. Penelitian ini bertujuan untuk menghasilkan mini teori pada pembelajaran berbasis bukti dengan menggunakan pendekatan teori APOS tentang struktur mental dan mekanisme mental siswa. Peran pembelajaran untuk meningkatkan kemampuan penalaran siswa. Metode penelitian yang digunakan adalah design research tipe validasi. Subjek penelitian sebanyak 34 orang siswa SMA Negeri 1 Palembang. Untuk itu penelitian ini akan membahas konstruksi HLT yang akan menjadi mini theory. Dalam tulisan ini berfokus kepada pembahasan HLT pembelajaran berbasis bukti dengan penggunaan pendekatan teori APOS pada pembelajaran pertama. Hasil dari penelitian ini adalah membandingkan HLT dengan ALT, mengembangkan pembelajaran pembuktian, dan menguji kebenaran dari hipotesis yang dibuat berdasarkan metodologi penelitian. Melalui penelitian ini siswa didapat memperoleh kemampuan penalaran, dan pembelajaran yang diberlangsungkan sudah sesuai dengan HLT yang dirancang.

Kata kunci: APOS; HLT; kemampuan penalaran; pembelajaran berbasis bukti.

Abstract

Proof-based learning is learning mathematics through proof and proving to strengthen students' concepts. The use of APOS theory (Action, Process, Object, and Schema) aims to describe students' mental structures summarized in Hypothetical Learning Trajectory (HLT). In the interest of learning mathematics, students require to have reasoning ability. This study aims to produce a mini theory of proof-based learning using the APOS theory approach about mental structure and mental mechanisms. The role of learning is to improve students' reasoning ability. The research method used is the design research validation type. The research subjects were 34 students from SMA Negeri 1 Palembang. For this discusses the construction of HLT which will become a mini theory. In this article will focus on the first lesson's HLT proof-based learning using APOS theory approach. This study's results are to compare HLT and ALT, develop proof learning, and test the truth of the hypotheses made based on the research methodology. Through this research students gain reasoning abilities, and the lesson that takes place is in accordance with the designed HLT.

Keywords*: APOS; HLT; proof-based learning; reasoning ability.*

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INTRODUCTION

Learning mathematics has a body, namely calculation, and a soul, namely analysis (Eriksson, Estep, & Johnson, 2004). Proof has a role in the soul of mathematics (Mañosa, 2022), therefore, proof can be one of the media for learning mathematics (Laamena, Nusantara, Irawan, & Muksar, 2018; Pfeiffer & Quinlan, 2015). The beauty of proving is when you finish and/or understand the proof (Pfeiffer & Quinlan, 2015; Wolchover, 2017). In the analysis of proof, it is necessary to have the ability to understand and relate proof to one another through argumentation (Ahmadpour, Reid, & Fadaee, 2019; Shinno et al., 2018). Proof is a unique series of logical arguments that make a statement true and need reasoning abilities for achieving it (Hanna & Reid, 2019).

According to Principles and Standards for School Mathematics NCTM (2000) said that reasoning is one of the five abilities that students must possess, namely problem-solving, reasoning, communication, connection, and representation. Proof-based learning can accommodate enhancing reasoning ability. Learning about proof will be able to improve students' analytical and logical thinking skills (Mañosa, 2022; Modeste et al., 2017). Learning about proof provides mathematical abilities that must be possessed by students (Kemendikbud, 2019). Proof learning experience is an important thing in mathematics because it supports the construction of thinking mathematically (Laamena et al., 2018).

There is some research on proof learning, namely Ahmadpour et al. (2019) about modelling the way students understand proof by reading proof for abstracting and formulating proof. The other research, Rocha (2019) gives suggestions for conducting research that teaches simple proof that many math teachers ignore. In China, there is research on developing teaching material about proof learning (Fan, et al., 2018), and also Zhang & Qi (2019) similar study about developing textbooks. Learning related to proof must be developed, one of the effective ways of learning is proof-based learning (Hanna & Reid, 2019; Shinariko, et al., 2020; Wittmann, 2021). But in learning proof students tend to give up because learning proof is too difficult (Selden & Selden, 2008). One way that is effective and will also be used in later learning is the use of two-column proof (Gemander et al., 2020).

APOS (*Action, Process, Object, Schema*) is a theory that focuses on the mental attitude of students during learning in constructing mathematical concepts (Arnon et al., 2014; Saftari et al., 2020). There is some research about APOS, namely Saftari et al. (2020) research on the development of student activities using APOS theory to understand the concept of Riemann sum. Mulyono (2011) discusses the role of APOS theory, detailing the indicators obtained from implementation in learning. Syamsuri & Marethi (2018) discusses the analysis of students' cognitive processes in proof activities. Gemander et al., (2020) and Syamsuri, et. al 2017) research explains why students are unable to construct proof using APOS theory in the analysis stage. Arnawa, et al. (2007) the study discusses the application of APOS theory to improve students' proofing skills. Wijayanti, et. Al (2019) discusses the mental structure of students in learning proof describe using APOS. Chamberlain & Vidakovic (2021) use APOS to analyze students' proof comprehension and understanding.

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Based on the research stated above, there have not been clearly stated how students' understanding about proof. The students' mentality in the implementation of proof learning has not appeared which will be related to their reasoning ability. Syamsuri & Marethi (2018) the discussion about the students' unable to prove were seen from the mental structure of APOS and provided suggestions for further research. From the problems stated above, the development of proof-based learning with the APOS theory approach will be made for enhancing students' reasoning ability.

METHOD

The research method used in this research is the type of design research validation. Design Research uses procedures from Gravemeijer (1994) and Gravemeijer & Cobb, 2006) to develop a design proof-based learning with APOS theory approach and produce a mini theory for learning. Therefore, the goal of developing a hypothetical learning trajectory (HLT) is to become a Mini Theory. An HLT consists of student learning objectives, math assignments that will be used to encourage student learning, and hypo-theses about student learning processes (Martin, Simon, & Tzur, 2012).

The research was conducted in SMA Negeri 1 Palembang with 34 students in grade 10 as subjects. There will be a more detailed explanation of HLT, by using purposive sampling (Rai, Alkassim, & Tran, 2015), the explanation will be more specific as a representation of other students. The data collected were class observation, learning videos, students' worksheets, and test results. The research cycle is divided into 3 stages (initial design, experiment, and retrospective analysis).

Preliminary design

At the initial stage, Hypothetical Learning Trajectory (HLT) and the learning instruments was made and improved time to time. The function of HLT is guidance for teachers for learning in class. HLT contains learning objectives, math ideas, and math activities. HLT contains a description of learning that exists in the APOS mental structure (Action, Process, Object, and Schema).

Experimentation

In the experimental stage, the HLT that has been designed will be tested on students. Tests are carried out to see whether the learning that has been made during the Preliminary design is according to the reality of the field. Everything that happens in the Experimentation stage will be an improvement/modification to the HLT design that has been made so it can be used in further research.

Retrospective analysis

The data analyze from observation, interview, and tests. In the analysis phase will focus on analyzing the application of mental structures that exist in the APOS theory (Action. Process, Object, and Schema). The description of proof-based learning with APOS theory in this study will be described as follows (Arnon et al., 2014; Syamsuri et al., 2017):

- Action: A transformation when it is a reaction to stimuli perceived as externally. Action indicators are what students can operate and prove.
- Process: An individual can repeat the action, and it has become the interior of the student's mental structure. The process indicator is that students can take action independently.

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- Object: If someone is aware of the process as a whole, then students will be aware of the transformation of the actions taken and can construct the transformation. The object indicator is that students can connect the information provided in a formal proof.
- Schema: Mathematical topics involving actions, processes, and objects that are organized and linked to the APOS framework. For example the connection between the concept of the exponential and the concept of radical form.

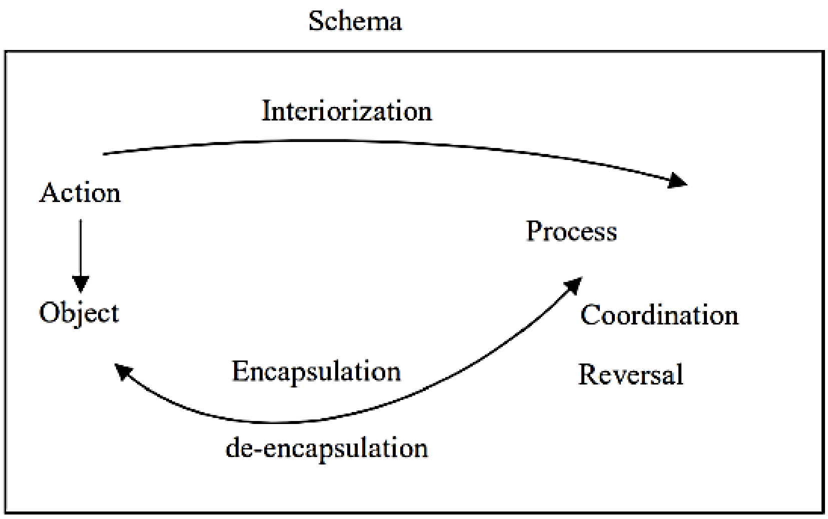

Figure 1. Mental structure and mental mechanism in APOS (Arnonet al., 2014)

RESULTS AND DISCUSSION

Preliminary design

This research will discuss Hypothetical Trajectory Learning (HLT) which was used in the first lessons. This lesson will focus on the exponential concept and divided into four activities. It was designed in proof-based learning with the APOS theory approach. The purpose of conducting this learning is to increase students' reasoning abilities. From the data collected and observed the video of learning during the field test to see if the learning that took place could support the teacher according to the original purpose. The results of students' worksheets will also be analyzed.

Using (Laamena et al., 2018; Reid & Vallejo Vargas, 2019; Shinariko et al., 2020), Learning design has been made in proof-based learning. The learning will focus on enhancing students' proof and reasoning abilities. Learning design will use simple proof (Rocha, 2019) dan worked-example (Kirsch, 1991; Laamena et al., 2018). Learning will also be guided by the use of APOS theory (Action, Process, Object, Schema) (Arnon, Cottrill, Dubinsky, Oktaç, Fuentes, et al., 2014; Mulyono, 2011; Syamsuri & Marethi, 2018; Wijayanti et al., 2019). The development of HLT and learning instruments that had been made were consulted with experts as preliminary design. The design had also already been tested first in the small group and got small changes and improvements and shown in Figure 2.

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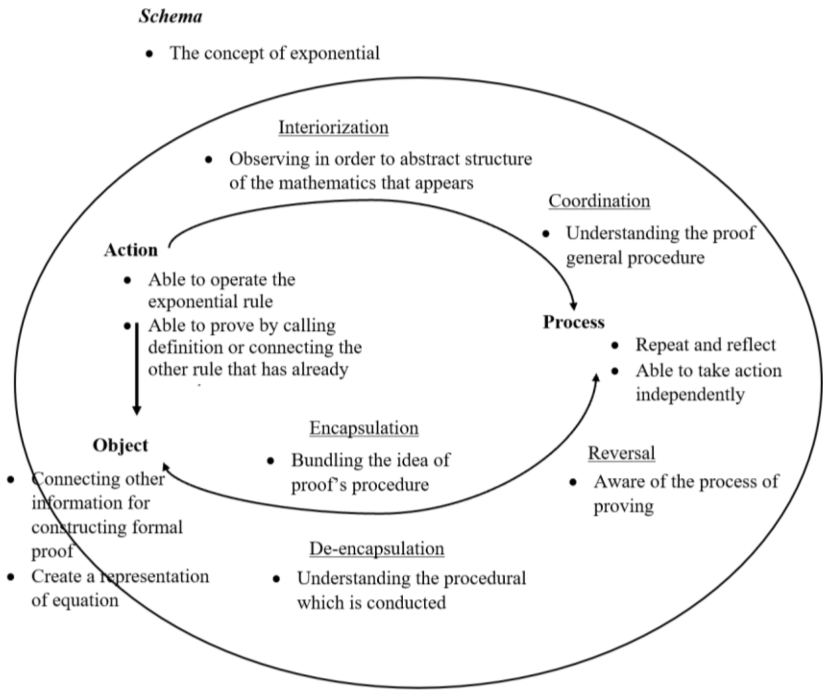

Figure 2. HLT proof-based learning using theory APOS

HLT which has been designed in Figure 2 becomes a guide for teachers in learning and will be compared with ALT. Conjectures or student guesses have also been made and will be discussed later. Before conducting the research, the researcher together with the model teacher had a forum discussion first. Discussions were conducted about the research flow, learning that will take place, and the comparison between HLT and ALT.

Experimentation

Class observations and pre-tests are carried out before the learning takes place. Observation aimed to find out the student's learning environment. For the pre-test for knowing students' initial ability. Interviews were also conducted with students using a semi-structured interview draft about students' knowledge of proof. From the results of interviews with 3 randomly selected people, it was stated that students had never done proof in previous learning. From the results of interviews with field test subjects and small group before, the researcher will add more lessons about simple proof. The goal is that students understand more about the proof procedures so the APOS Schema can be accomplished.

For the next meeting, the learning that had been designed was carried out. Before starting the lesson, the teacher divides the students into groups of two people. Then the worksheets will be distributed to students and the lesson will begin. The Lesson is divided into four activities, namely the exponent of positive integers, multiplication between exponent, the power of exponent, and the division of exponential.

Retrospective analysis

The Exponent of Positive Integers

In this section, firstly, the teacher will explain the definition of the exponential form, namely $a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{factor\ n}$ and also an explanation of the generalization. In this section, the

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explanations of explanation of exponent also include prime factorization. Some students have difficulty converting something into prime factorization form on large numbers. So the teacher will explain about it:

$$3.600 = 36 \cdot (10) \cdot (10) \dots (1)$$
$$= 6 \cdot 6 \cdot (2 \cdot 5) \cdot (2 \cdot 5) \dots (2)$$
$$= 2^4 \cdot 3^2 \cdot 5^2 \dots (3)$$

Students are also asked to work on similar questions so that they can perform the Process stage (repeat and reflect) so that they become student interiors (Arnon, Cottrill, Dubinsky, Oktaç, Fuentes, et al., 2014; Mulyono, 2011). The definition and understanding that had been learned will be useful for the next activities which contain more proof.

Multiplication between Exponent

In this section, we use the theory from Selden & Selden (2008) and Wittmann (2021) which states that generalization proofs can help students to understand more in-depth proof procedures. The purpose of generalization proof is so that students can better understand simple proof later (Rocha, 2019). After that, students were asked to prove the exponential property: $a^p \times a^q = a^{p+q}$. Students are led to prove using the definitions that have been explained.

Translate:
Prove the following exponential properties!
$$a^p \times a^q = a^{p+q}$$

So $a^p \times a^q$ alike, then a as base and has p factor and a as the base multiplied with q factor

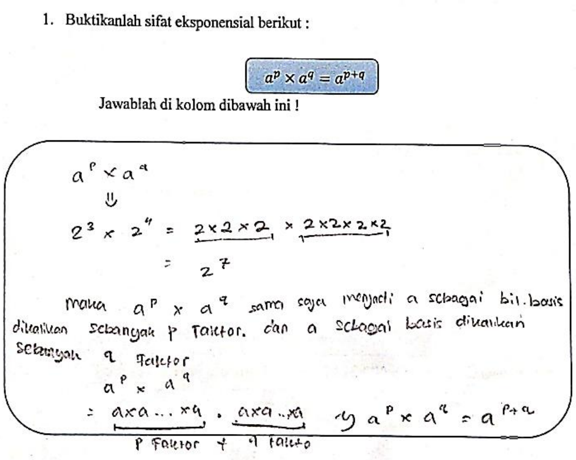

Figure 3. The Action of DNF dan SKR for proving $a^p \times a^q = a^{p+q}$

Through the Action shown in Figure 3, it can be seen that DNF and SKR did the proof correctly, but they also used generalizations for conceptualization first. DNF and SKR argue with sentences that are quite difficult to understand in completing the proof. But in the proof strategy is correct, it makes the action that happened become the object. The arguments that students stated are hard to understand, the Encapsulation dan Reversal of the Process have not appeared similar to those (Wijayanti et al., 2019) research.

Meanwhile, other groups who answer directly using definitions, and some groups are still facing difficulties. After the work is done by the students, the teacher discusses the proof in class. For students who are still confused, the teacher only asks students to observe the work of the proof in the hope that it can become a student interior (Arnawa et al., 2007). Students who still find it difficult to prove will be given new ideas, namely proof of $a^p \cdot a^p = a^{2p}$:

$$a^p \cdot a^p$$

$$a^p \cdot a^p$$

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$$= \underbrace{a \cdot a \cdot \ldots \cdot a}_{factor\, p} \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{factor\, p} \ldots (1)$$

$$= \underbrace{a \cdot a \cdot \ldots \cdot a}_{factor\, p+p=2p} = a^{2p} \ldots (2)$$

This is one of the simple proofs taught by the teacher. The goal is for students can understand the proving procedure more deeply (Rocha, 2019). After doing the proof, students are also given practice questions to strengthen their concept of multiplication between positive integers.

The power of Exponent

Learning the power of positive integers will be linked to the previous two sub-chapters so that Objects can be achieved and also can make encapsulation happen. In this section, students are asked to prove to practice their proofing skills. In this section, the worked-example proof about generalization is less effective, because it makes students think wrongly in a formal proof. If students are not directed and given a clear understanding between formal proof and generalization proof, then the error will continue (Hanna, 2020).

Translate:
Prove the following exponential properties!

$(a^P)^q$: "a^P * " spread out as much as q factor, then added the factor that spread out

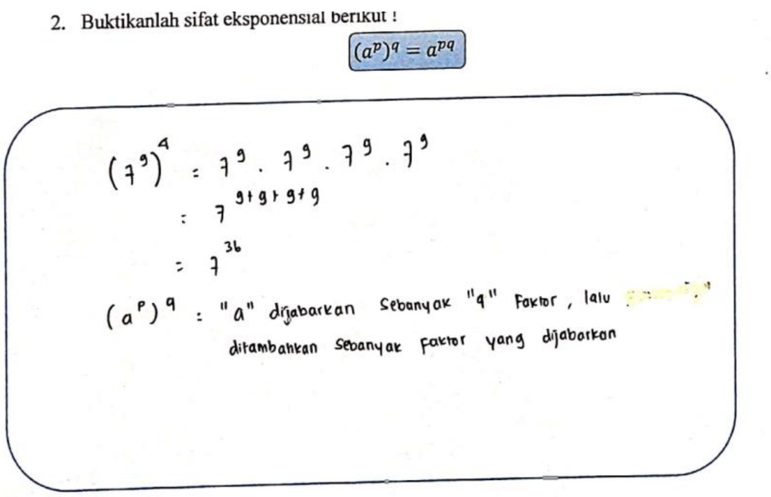

Figure 4. The generalization and argument of students

As in Figure 5, students are asked to prove but AS and OE students use generalization methods to prove it. It means students understand the definition of learning in the previous activities and they can apply it. But this is not enough to be called formal proof, but the arguments given are correct. The object would not happen if the students weren't able to prove it formally. The learning design is the lack of prior knowledge to students about proof deeply and its meaning, especially on the difference between formal proof and general proof.

The Division of Exponential

In this activity, students are introduced to the two-column proof. Students are asked to provide reasons for the proof that has already been proved. For students who are lacking, this two-column proof step will help them to more fully realize the proof procedure (Gemander et al., 2020). The two-column proof design is made for students who do not understand proof and can also participate in the learning process. The steps of proof have been made in the provided two-column proof and students only need to argue about what steps are used in the proof.

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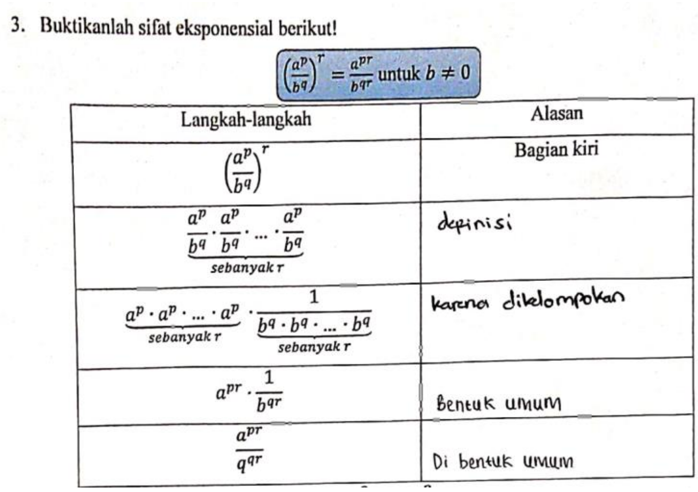

3. Buktikanlah sifat eksponensial berikut!

$\left(\frac{a^p}{b^q}\right)^r = \frac{a^{pr}}{b^{qr}}$ untuk $b \neq 0$

Langkah-langkah	Alasan
$\left(\frac{a^p}{b^q}\right)^r$	Bagian kiri
$\underbrace{\frac{a^p}{b^q}\cdot\frac{a^p}{b^q}\cdot\ldots\cdot\frac{a^p}{b^q}}_{\text{sebanyak } r}$	depinisi
$\underbrace{a^p \cdot a^p \cdot \ldots \cdot a^p}_{\text{sebanyak } r} \cdot \frac{1}{\underbrace{b^q \cdot b^q \cdot \ldots \cdot b^q}_{\text{sebanyak } r}}$	karena dikelompokan
$a^{pr} \cdot \frac{1}{b^{qr}}$	Bentuk umum
$\frac{a^{pr}}{q^{qr}}$	Di bentuk umum

Translation:

Reasoning
- Left side
- Definition
- Because grouped
- General form
- General form

Figure 5. Students reasoning in two-column proof

Figure 5 shows the work of ABAP and DPP students on two-column proof. They write down the reasons for the proof in brief. The purpose of using two-column proof is so that de-encapsulation in the APOS mental process, that is, students can clearly understand the proof procedure. Students can also provide reasons that relate to previous learning. This action has become an object for students because they can connect and understand.

Comparison between HLT dan ALT

The explanation of comparison between HLT and ALT will be started by showing mental structure in the first lesson. This result can be seen in Figure 6. Based on the result in Figure 6, from four activities, the HLT shows that Action, Process, and Object have emerged. However as shown in Figure 6, the Reversal has not appeared clearly, this is because the concepts in the proof have not been fully understood by some students. The next lesson will be shown the use of Reversal for strengthening students' concepts in proof. The encapsulation has not appeared in this lesson either. Encapsulation is achieved when students can bundle the idea of proof's procedure, in this case, the properties of proof. This meeting has already been designed for showing Encapsulation, but because students are not familiar with proof, this first lesson does not focus on that.

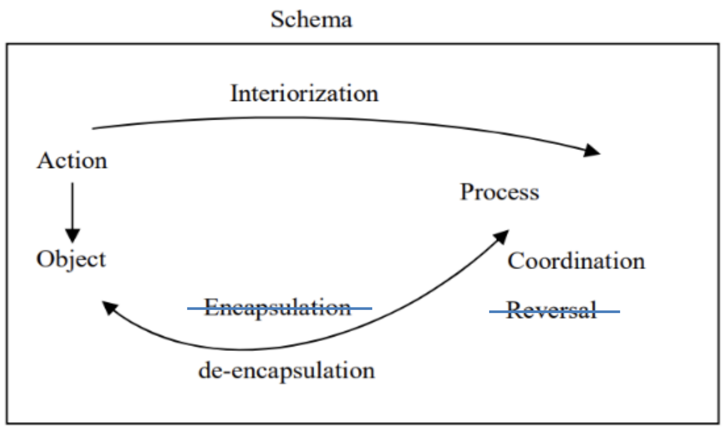

Figure 6. Mental structure in first lesson

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In the process stage, only a few highly capable students can do the proof independently (Arnon et al., 2014). Moreover, students are new to the proving concept, so firstly students' interest must be encouraged (Selden & Selden, 2008). At this meeting, the scheme only appeared partly to happen on the concept of exponents and had not linked with other learning. In the next lesson, the radical form had already design that showing Encapsulation, reversal, and will be made sure that the HLT design will be useful.

CONCLUSION AND SUGGESTION

From the HLT, in this paper, we focus on the first lesson namely exponential. In this lesson, there are 4 activities conducted. The HLT design for proof-based learning using APOS theory approach was compared with ALT. The mental structure in the first lesson is not showing Encapsulation and Reversal. For the next lesson's HLT design for fulfilling the mental structure that had not appeared. Students' mental structure was shown detailed for their reasoning ability. Therefore, the lesson conducted affected students in the reasoning ability. The lesson designed had been already according to HLT.

The study suggests that more research about proof is conducted in junior high school, so the students in high school at know about proof. In some lessons, students in junior high school can experience proof. For the next research, students will be given more simple proof and worked-example for learning purposes. Because students' reasoning ability enhance by proof-based learning.

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