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ANALYSIS OF OPTIMUM CONTROL ON THE IMPLEMENTATION OF VACCINATION AND QUARANTINE ON THE SPREAD OF COVID-19

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Abstract. *This study constructs an SVIR-type COVID-19 spread model into a model with control variables or optimum control problems. In the formulation of the model with controls, we set four control variables, namely vaccination strategy, quarantine, reduction of vaccine shrinkage, and treatment. Pontryagin 's maximum principle is applied in the model as a sufficient condition to achieve optimum conditions for minimizing the objective function* $J(X(\cdot), U(\cdot))$. *This study uses a numerical solution to describe the theoretical results. The results showed that the control model could accelerate the decrease in the number of individuals in the infected population class. We found that vaccination is a top priority that needs to be done to reduce the number of cases of COVID-19 infection. In addition, the implementation of quarantine can also be considered to accelerate the decrease in the number of individuals infected with COVID-19.*

Keywords: *COVID-19, optimum control, quarantine, treatment., vaccination*

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1. INTRODUCTION

Since its first appearance in Wuhan City (China), the coronavirus or known as Coronavirus Disease-2019 (COVID-19) has shocked people all over the world [1]. This virus can survive on solid objects for several hours according to ambient temperature conditions [2]. This makes it easy for COVID-19 to spread and infect every susceptible individual.

A person who has been infected with COVID-19 can initially experience clinical symptoms such as fever, cough, loss of smell, and shortness of breath. The time taken from the onset of infection until clinical symptoms appear is 14 days. If the patient is not treated immediately, it will result in death for the patient [3]. However, in some cases reported there are also infected individuals without clinical symptoms.

Several efforts to implement health protocols by the government to tackle the spread of the COVID-19 outbreak, ranging from large-scale social restrictions (PSBB) to the implementation of restrictions on community activities (PPKM), policies on the use of masks, and checking body temperature when entering areas or locations of public facilities. In addition, the government has also implemented a COVID-19 vaccination policy which began on January 13, 2021. This vaccination program has been running for one year, but cases of COVID-19 infection in Indonesia are still occurring. Nevertheless, efforts by the government and all levels of Indonesian society continue to be made to reduce the number of infection cases and stop the spread of COVID-19. No exception from academics, especially mathematicians.

Mathematical practitioners develop mathematical models to study the nature, dynamics, and effects of the spread of COVID-19. Mathematical models continue to be developed by the development of the characteristics of COVID-19 found in the field, both in terms of the pattern of spread and ways to anticipate its transmission. Several mathematical models of the spread of COVID-19 that have been developed can be found in [4]–[8]. The analysis of some of these models will provide an overview of the dynamics or the number of infection cases in the future based on mathematical rules. So that countermeasures or anticipatory steps can be taken earlier.

The model developed by [4] divides the human population into three subpopulation classes, namely: suspected (S), infected (I), and recovered (R). These four populations each represent a class of susceptible, infected, and cured individuals. One of the assumptions of the model built is that every individual who has recovered from COVID-19 will not be infected again a second time. In addition, this model has also considered the parameters of vaccination, as an effort to prevent the spread.

Next, we reconstructed the mathematical model of the spread of COVID-19 by adding the vaccinated population class (V) which can be seen in [9]. Individuals in the vaccinated population class state that individuals who have received the vaccine but still have the possibility of being infected with COVID-19 and at a certain time will be susceptible to the virus again. The model with the vaccinated population is constructed based on the facts found in the field that, individuals who have received the vaccine still have the possibility of being infected with COVID-19 [10]. Then the stability of the model is analyzed along with the level of sensitivity of the parameters to the basic reproduction number of the model. However, the completion of this model is still based on the assumption that all parameter values are constant. So in this study, the model is constructed into a model with optimum control or control problems. Several models of the spread of COVID-19 that apply the optimum control problem can be seen in [11]–[18].

The control model in this study has four control variables consisting of vaccination strategy, quarantine, reduction of vaccine shrinkage, and treatment. This control variable is the development of the previous model parameters which were declared constant. We apply the Pontryagin maximum principle as a necessary and sufficient condition to obtain the optimum conditions for the optimum control problem [19], [20]. In solving the optimum control problem, we make a numerical simulation as an illustration of the theoretical results that can be taken into consideration in setting policies to suppress the number of COVID-19 infection cases.

2. RESEARCH METHOD

The mathematical model of the spread of COVID-19, the SVIR type, represents the process of virus transmission by involving the administration of vaccines to a class of susceptible individuals. We calculated with the consideration that the efficacy of vaccines in forming antibodies could at any time decrease, and someone who has been vaccinated still has the potential to be infected with COVID-19.

This study divides the human population into four subpopulation classes, each susceptible population class (S), the vaccinated population class (V), the infected population class (I), and the recovered population class from COVID-19 (R). The total population is calculated by adding up all individuals in each population class, which is expressed by $\boldsymbol{N = S + V + I + R}$. The dynamics of the human population in this model begin with the birth of new individuals who enter the vulnerable population class at a rate of $\boldsymbol{\Lambda}$. Some of the individuals in the vulnerable population class will be vaccinated and enter the vaccinated population class at a rate of $\boldsymbol{\tau}$. Individuals in the vaccinated population class are less likely to contract COVID-19 than individuals in the susceptible population class. This depends on the level of effectiveness of the vaccine given. In this model, we formulate the rate of decline in vaccine effectiveness as $\boldsymbol{\sigma}$. Individuals in the vaccinated population class at any time can also be re-entered into the susceptible population class if the vaccine antibody in the body decreases at a rate of $\boldsymbol{\eta}$.

Transmission of COVID-19 will occur if individuals in the susceptible population class and the vaccinated population class each come in contact with individuals in the infected population class. This resulted in the entry of individuals in the susceptible population class and individuals in the vaccinated population class respectively into the infected population class at a rate equal $\boldsymbol{\beta}$to for susceptible individuals and $\boldsymbol{\beta\sigma}$ for vaccinated individuals. Individuals in the infected population class have the possibility of recovering through treatment at a rate of $\boldsymbol{\theta}$, recovering naturally at a rate of $\boldsymbol{\gamma}$, or dying from disease at a rate of $\boldsymbol{\xi}$. Every individual who recovers naturally or through treatment will be included in the cured population class. Individuals in the recovered population class are assumed not to be re-infected. All individuals in each population class have a probability of dying at a rate of $\boldsymbol{\mu}$.

Furthermore, in this model, we define control variables $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\boldsymbol{u_3}$, and $\boldsymbol{u_4}$ which each represent a vaccination strategy to increase antibodies, a quarantine strategy to reduce contact with individuals in the infected population class, a vaccine shrinkage reduction strategy to maintain antibodies, and a treatment strategy to increase the number of infected individuals. individuals in the population class recovered. All control variables are dynamic and depend on changes in time.

Based on the explanation above, a system of differential equations is formulated which is a model for the spread of COVID-19 type SVIR with the following controls:

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda + \eta(1-u_3)V - \frac{\beta(1-u_2)I\,S}{N} - (u_1+\mu)s \\
\frac{dV}{dt} &= u_1 S - \frac{\sigma\beta(1-u_2)I\,V}{N} - (\eta(1-u_3)+\mu)V \\
\frac{dI}{dt} &= \frac{\beta(1-u_2)I\,S}{N} + \frac{\sigma\beta(1-u_2)I\,V}{N} - (\gamma+\mu+\xi+u_4)I \\
\frac{dR}{dt} &= (u_4+\gamma)I - \mu R
\end{aligned}
\tag{1}
$$

To obtain optimum results in suppressing the spread of COVID-19, we used Pontryagin's maximum principle in model completion.

3. RESULTS AND DISCUSSION

3.1. Optimum Control Problem

The optimum control problem which is the main focus of this research is to determine the control function $U = \{u_i \mid i = 1,2,3,4\}$ to achieve the desired goal. The objective function in this study is formulated as follows:

$$
J\big(X(\cdot),U(\cdot)\big) = \int_0^T \left(A\,I + \frac{B_1}{2}u_1^2 + \frac{B_2}{2}\,u_2^2 + \frac{B_3}{2}u_3^2 + \frac{B_4}{2}u_4^2\right)dt \tag{2}
$$

where A states the weights of individuals in the infected population class, while B_1, B_2, B_3, and B_4 state the weights for the parameters that are subject to control. Meanwhile $\frac{B_1}{2}u_1^2$, $\frac{B_2}{2}u_2^2$, $\frac{B_3}{2}u_3^2$, and $\frac{B_4}{2}u_4^2$, respectively, state the costs of vaccination, quarantine, reduction of vaccine depreciation, and treatment. In addition, the square of each control variable shows that the relationship between costs incurred on each control variable has a nonlinear relationship with the number of cases of COVID-19 infection. In this study, costs are assumed to be limited, so the control function is $u_i (i = 1,2,3,4)$ limited according to

$$\mathcal{U} = \{U(\cdot) \in L^\infty([0,T]; \mathbb{R}^4 | 0 < u_i \le u_{i\,max} < 1), \forall t \in [0,T]\} \tag{3}$$

Define a set $\mathcal{X} = \{X(\cdot) \in W^{1,1}([0,T]; \mathbb{R}^4) | X(\cdot)\}$ where $X = (S, V, I, R)$, then the optimum control problem to be solved is to bring the state into an optimum condition $X^*(\cdot) = \left(S^*(\cdot), V^*(\cdot), I^*(\cdot), R^*(\cdot)\right) \in \mathcal{X}$ at the time interval $[0,T]$ by minimizing the objective function which can be written as follows:

$$J\left(X^*(\cdot), U^*(\cdot)\right) = \min_{X(\cdot), U(\cdot) \in \mathcal{X}\times\mathcal{U}} J\left(X(\cdot), U(\cdot)\right) \tag{4}$$

as well as the constraint function:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda + \eta(1-u_3)V - \frac{\beta(1-u_2)I\,S}{N} - (u_1+\mu)s \\
\frac{dV}{dt} &= u_1 S - \frac{\sigma\beta(1-u_2)I\,V}{N} - (\eta(1-u_3)+\mu)V \\
\frac{dI}{dt} &= \frac{\beta(1-u_2)I\,S}{N} + \frac{\sigma\beta(1-u_2)I\,V}{N} - (\gamma+\mu+\xi+u_4)I \\
\frac{dR}{dt} &= (u_4+\gamma)I - \mu R \\
S(0) &\ge 0, V(0) \ge 0, I(0) \ge 0, R(0) \ge 0.
\end{aligned} \tag{5}$$

Based on equations (2) and (5) the Hamilton Function is defined as follows:

$$\begin{aligned}
H(X,U,\lambda) = {} & A\,I + \frac{B_1}{2}u_1^2 + \frac{B_2}{2}u_2^2 + \frac{B_3}{2}u_3^2 + \frac{B_4}{2}u_4^2 \\
& + \lambda_1\left[\Lambda + \eta(1-u_3)V - \frac{\beta(1-u_2)I\,S}{N} - (u_1+\mu)s\right] \\
& + \lambda_2\left[u_1 S - \frac{\sigma\beta(1-u_2)I\,V}{N} - (\eta(1-u_3)+\mu)V\right] \\
& + \lambda_3\left[\frac{\beta(1-u_2)I\,S}{N} + \frac{\sigma\beta(1-u_2)I\,V}{N} - (\gamma+\mu+\xi+u_4)I\right] \\
& + \lambda_4[(u_4+\gamma)I - \mu R].
\end{aligned}$$

By applying Pontryagin's maximum principle, we get the adjoin function

$$\begin{aligned}
\frac{d\lambda_1^*}{dt} &= \lambda_1^*(u_1+\mu) + \frac{I(\lambda_1^*-\lambda_3^*)(1-u_2)\beta}{(S+V+I+R)} - \lambda_2^* u_1; \\
\frac{d\lambda_2^*}{dt} &= \lambda_2^*(\eta - \eta u_3 + \mu) + \frac{(\lambda_2^*-\lambda_3^*)(1-u_2)\beta\sigma I}{S+V+I+R} - \lambda_1^*(1-u_3)\eta; \\
\frac{d\lambda_3^*}{dt} &= \lambda_3^*(u_4+\gamma+\mu+\xi) - \frac{(1-u_2)[(\lambda_3^*-\lambda_2^*)\sigma V + (\lambda_3^*-\lambda_1^*)S]\beta}{S+V+I+R} - \lambda_4^*(u_4+\gamma) - A; \\
\frac{d\lambda_4^*}{dt} &= \lambda_4^*\mu,
\end{aligned} \tag{6}$$

and control function

$$u_i^* = \max[0, \min(\tilde{u}_i, u_{i\,max})]$$

or

$$u_i^* = \begin{cases} 0 & ; \quad \tilde{u}_i \le 0 \\ \tilde{u}_i & ; \quad 0 < \tilde{u}_i \le u_{i\,max} \\ u_{i\,max} & ; \quad \tilde{u}_i > u_{i\,max} \end{cases}$$

where:

$$\tilde{u}_1 = \frac{(\lambda_1^* - \lambda_2^*)S^*}{B_1},\ \tilde{u}_2 = \beta \frac{I^*[(\lambda_3^* - \lambda_2^*)\sigma V^* + \lambda_3^* S^* - \lambda_1^* S^*]}{B_2(S^* + V^* + I^* + R^*)}$$
$$\tilde{u}_3 = \frac{(\lambda_1^* - \lambda_2^*)\eta V^*}{B_3}, \qquad \tilde{u}_4 = \frac{(\lambda_3^* - \lambda_4^*)I^*}{B_4}.$$

Since $S(T), V(T), I(T)$ and $R(T)$ are arbitrary, the optimum condition must satisfy the following transversality conditions:

$$\lambda_1(T) = 0, \lambda_2(T) = 0, \lambda_3(T) = 0, \lambda_4(T) = 0.$$

3.2. Numerical Simulation

Equations (5) and (6), each of which *state* and *co-state* equations, are in the form of a system of nonlinear differential equations. Therefore, a numerical approach is needed to obtain a solution to the equation. In this study, we use the Runge-Kutta approach of order 4 with a forward scheme for solving *state equations* and a backward scheme for solving *co-state equations*. The initial value of each population model used in this study is $S(0) = 268.777.880$, $V(0) = 0$, $I(0) = 126.313$, $R(0) = 695.807$. The parameter values can be seen in Table 1.

Table 1 . Parameter value

Parameter	Symbol	Mark	Source
Birth rate	Λ	$2{,}4 \times 10^6$/year	[21]
Natural death rate	μ	$6{,}25 \times 10^{-3}$/day	[9]
Vaccine shrinkage rate	η	0,0027/day	[9]
Infection rate	β	0,084/day	[9]
Rate of decline in vaccine effectiveness	σ	0,6	[9]
The death rate due to COVID-19	ξ	$2{,}2114 \times 10^{-4}$/day	[9]
Natural healing rate	γ	$1{,}042 \times 10^{-3}$/day	[9]

The calculation results show that over time, the number of susceptible individuals has decreased from the initial number, as shown in Figure 1. The number of vaccinated individuals continues to increase from the original number. However, there was a difference between the number of individuals who were given control and the number of individuals without control. The decrease in the number of susceptible individuals who were given control was greater than the number of susceptible individuals without being given control. In addition, it can be seen that the number of vaccinated individuals who were given control experienced a greater increase compared to the number of individuals vaccinated without being given control. This shows that vaccination strategies and reducing vaccine shrinkage are optimum in reducing the potential for transmission or spread of COVID-19.

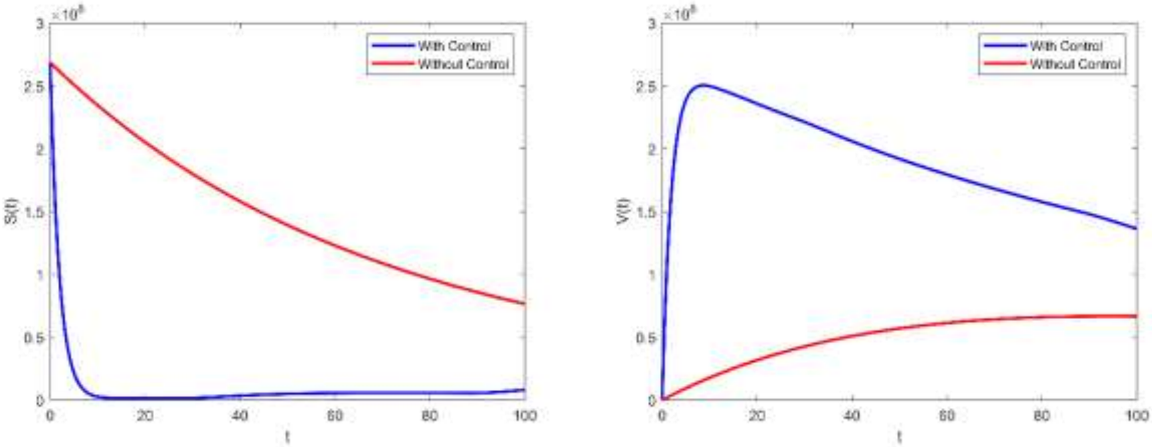

Figure 1. Dynamics of Vulnerable Population Class and Vaccinated Population Class

In addition, there are also differences in the results of giving control and without control between the number of infected individuals and recovered individuals. Figure 2 shows that, without control, the number of infected individuals increased, followed by an increase in the number of recovered individuals. However, the number of infected cases is still higher than the number of recovered cases. The provision of control can reduce the number of infected individuals until finally there are no more individuals infected with COVID-19. The decline in infection cases was followed by a decrease in the number of recovered individuals, due to the reduced number of individuals requiring treatment due to contracting COVID-19. This shows that quarantine and treatment strategies are optimum in reducing the number of COVID-19 cases until they reach a condition where there are no additional cases of infection.

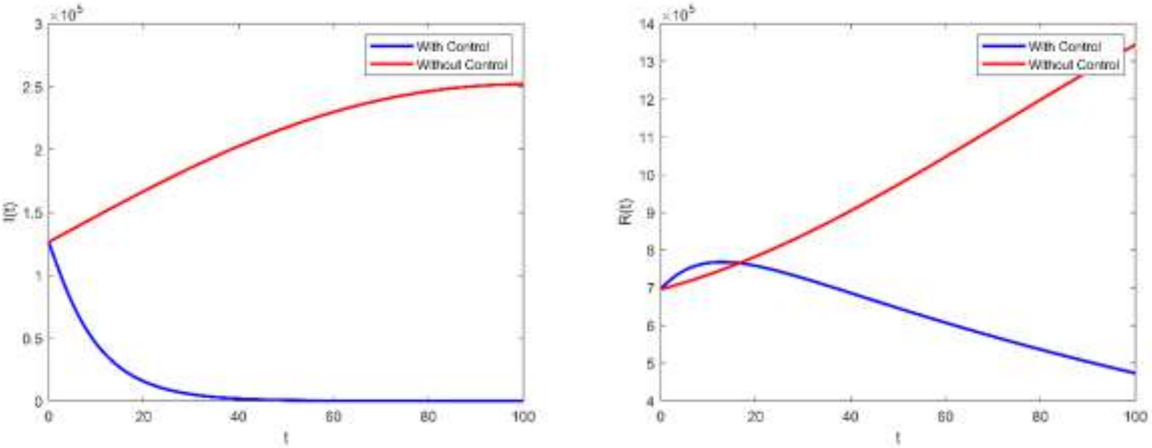

Figure 2. Dynamics of Infected Population Class and Cure Population Class

Furthermore, the calculation is carried out by applying three conditions to the model with control. The three conditions consisted of the use of a model with all control strategies, a model without a vaccination strategy, and a model without a quarantine strategy. Based on Figure 3, the three conditions applied still have an impact in the form of a decrease in the number of individuals in the infected population class. However, without a vaccination strategy, this decline was slower than for the other two conditions. The results of this calculation indicate that the vaccination strategy can be made a top priority in handling cases of the spread of COVID-19 cases. Nevertheless, a quarantine strategy is still needed to accelerate the decline in the number of individuals infected with COVID-19.

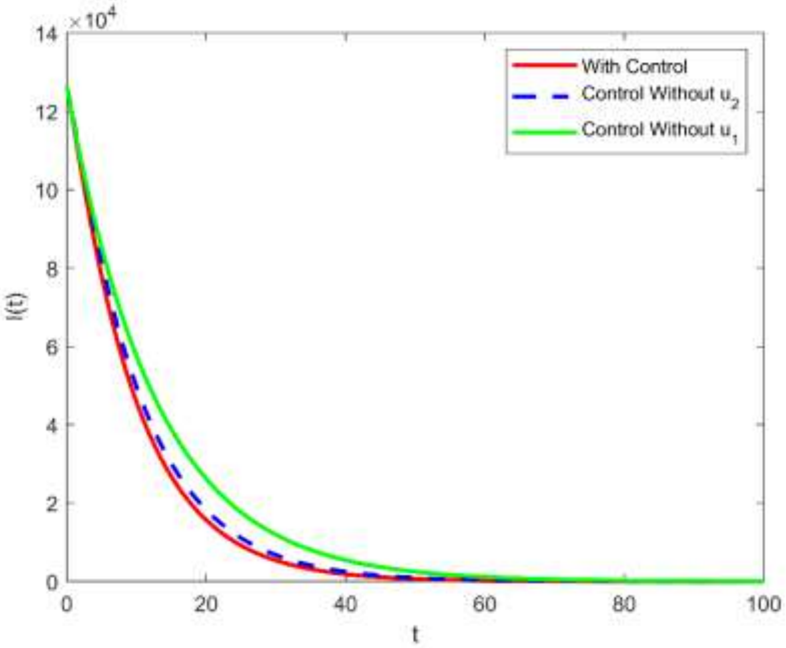

Figure 3. Dynamics of Infected Population Class Based on the Effect of Vaccination and Quarantine

The final part of the numerical simulation explains how the steps must be taken to obtain the optimum control strategy. Due to high medical costs and vaccine limitations, we set the maximum limit of the vaccination strategy and reduce vaccine shrinkage to 0.5 per day. Based on Figure 4, vaccination strategies and reducing vaccine shrinkage need to be carried out optimumly. If at the beginning of time each strategy can reach its maximum value, after 10 days running, both strategies can be periodically reduced which will later help in saving costs. Quarantine strategies need to be implemented from the start of the spread of the disease, at a rate of 0.58 per day. After 90 days, the quarantine can be lowered gradually until there are no more cases of COVID-19 infection. The same thing applies to the treatment strategy which has a constant rate of 0.5-8 per day and is then stopped until no more individuals infected with COVID-19 are found in the population.

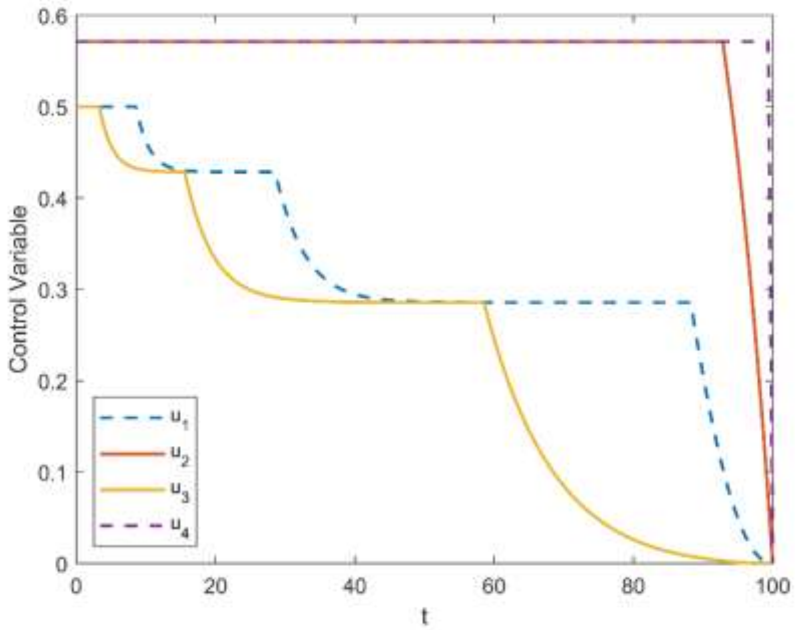

Figure 4. Dynamics of Vaccination Strategy, Quarantine, Vaccine Depreciation Reduction, and Treatment

4. CONCLUSIONS

The application of a controlled model may help reduce the number of cases of COVID-19 infection which is characterized by no more individuals in the infected population class at the end of the period. In addition, the implementation of the four control strategies needs to be carried out maximally at the beginning so that over time each strategy can be relaxed or reduced gradually. This can help save costs in the COVID-19 spread control program.

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