H_∞ Control of Polynomial Fuzzy Systems: A Sum of Squares Approach

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Abstract. This paper proposes the control design of a nonlinear polynomial fuzzy system with H_∞ performance objective using a sum of squares (SOS) approach. Fuzzy model and controller are represented by a polynomial fuzzy model and controller. The design condition is obtained by using polynomial Lyapunov functions that not only guarantee stability but also satisfy the H_∞ performance objective. The design condition is represented in terms of an SOS that can be numerically solved via the SOSTOOLS. A simulation study is presented to show the effectiveness of the SOS-based H_∞ control design for nonlinear polynomial fuzzy systems.

Keywords: *H_∞; nonlinear; polynomial fuzzy controller; polynomial fuzzy system; polynomial Lyapunov function; SOSTOOLS; sum of squares.*

1 Introduction

A Takagi-Sugeno (T-S) based fuzzy control system [1] is a type of fuzzy control that has a systematic structure. A T-S fuzzy model was used in designing such a control system. A set of fuzzy rules was used to represent the global nonlinear system in the form of a set of local linear models. As an alternative approach to describe complex nonlinear systems, the T-S fuzzy model drastically reduces the number of rules in the modeling of higher order nonlinear systems. In twenty years, studies of T-S fuzzy modeling (e.g., [2,3]) have provided the foundation for development of systematic approach to stability analysis and control design of fuzzy control systems. However, since the representation of the consequent parts is limited to a linear model, system modeling is not flexible.

A nonlinear system can be modeled by a polynomial fuzzy model. A polynomial fuzzy modeling and control framework is a generalization of the T-S fuzzy model and is more effective in representing nonlinear control systems [4].

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A general block diagram of the polynomial fuzzy control system is depicted in Figure 1. The real system is represented by a polynomial fuzzy model. This polynomial fuzzy model is used to design a polynomial fuzzy controller. The design of controllers to achieve the synthesis objective is performed through parallel distributed compensation (PDC) [5]. The main idea of PDC is to derive each control rule so as to compensate each rule of the polynomial fuzzy model. The polynomial fuzzy controller shares the same fuzzy sets with the polynomial fuzzy model in the premise parts. Modern control techniques can be extended to analyze the stability of a polynomial fuzzy model and to design a polynomial fuzzy controller. The final output of the polynomial fuzzy controller is obtained by a fuzzy blending of each individual feedback gain.

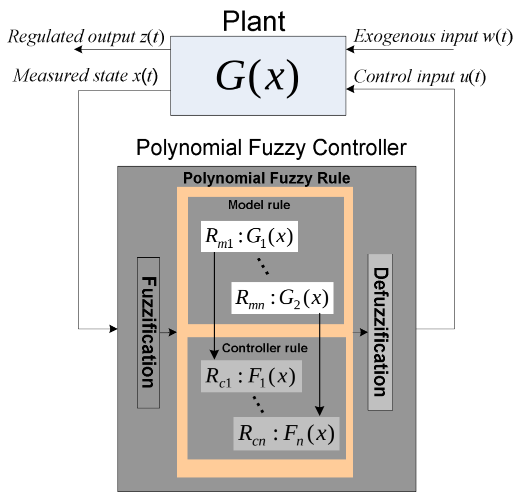

Figure 1 Polynomial fuzzy control system.

Research on stability analysis and control design of polynomial fuzzy systems has been conducted (e.g., [6,7]), but in these studies, designed controllers still have a weakness in that the robustness of the closed-loop system is not guaranteed.

A sum of squares (SOS) is a multivariate polynomial that can be written as a sum of squares of other polynomials. An SOS is globally non negative, restricting polynomials to be an SOS implies their positive semidefiniteness [8]. This property is important in many control applications, where we can replace various polynomial inequalities with SOS conditions. An SOS program is an optimization problem with SOS constraints. Studies of SOS programs in systems and control theory have been conducted in fields such as nonlinear stability and control synthesis [9], and state feedback and output feedback control [10]. This paper presents a sum of squares (SOS) approach for H_∞ control of a nonlinear system using polynomial fuzzy systems. Stability analysis and control design for polynomial fuzzy systems is represented in terms of an SOS that can be numerically solved via SOSTOOLS [11].

In the real world, various forms of uncertainties exist in the implementation of a control system, for instance plant parameters, sensor noise, and plant disturbance. In order to keep the system working satisfactorily, we have to make it robust, in other words, insensitive to such uncertainties. In the past decade, design methods of robust control based on the H_∞ norm have been conducted (see for instance, [9,12-14]). The objective of H_∞ control is to find a controller u such that the closed-loop system is asymptotically stable and L_2 gain from the exogenous input w to the objective signal to be regulated z is less than or equal to a γ value. This will provide a disturbance attenuation level of γ. The H_∞ control design may also be applied to guarantee robustness with respect to unstructured dynamic uncertainty [15].

Preliminary results of this paper appeared in [16], but were restricted only to derivation of the main design theorem. The present paper extends [16] by providing a numerical example and simulation results.

The rest of this paper is organized as follows. In Section 2, this paper briefly reviews the polynomial fuzzy model. Section 3 presents the design of a stable polynomial fuzzy control and Section 4 presents the design of a polynomial fuzzy system with H_∞ control performance objective. A numerical example is presented in Section 5. Finally, the conclusion is drawn in Section 6.

2 Polynomial Fuzzy Model

In this paper we consider a nonlinear system represented by a polynomial fuzzy model. The main difference between a T-S fuzzy model [1] and a polynomial fuzzy model [4] is the representation of the consequent part. A T-S fuzzy model consequent is represented by a linear model, while a polynomial fuzzy model consequent [4] is represented by a polynomial model such as shown in (1).

$$\textit{If } a_1(t) \textit{ is } M_{i1} \textit{ and ...and } a_1(t) \textit{ is } M_{ip}$$

$$\textit{Then } \dot{x}(t) = A_i(x(t))Z(x(t)) + B_i(x(t))u(t), \; i = 1, ..., r. \tag{1}$$

where $a_j(t)(j=1,2,...,p)$ is the premise/antecedent variable. M_{ij} is the membership function associated with the *i*-th model rule and the *j*-th premise variable component. $A_i(x(t))$ and $B_i(x(t))$ are polynomial matrices in $x(t)$. The column vector whose entries are all monomial in $x(t)$ is denoted by $Z(x(t))$. $Z(x(t)) \in R^N$ is an $N \times 1$ vector of a monomial in $x(t)$. A monomial in $x(t)$ is a function of the form $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, where $\alpha_1, \alpha_2, ..., \alpha_n$ is a nonnegative integer.

In this paper, it is assumed that $Z(x(t)) = 0$ iff $x(t) = 0$.

The overall polynomial fuzzy model is obtained by fuzzy blending of each polynomial model equation in the consequent part. By using the weighted average of each rule's output, the defuzzification process of model (1) can be represented as

$$\dot{x}(t) = \frac{\sum_{i=1}^{r} \varpi_i(a(t))\{A_i(x(t))Z(x(t)) + B_i(x(t))u(t)\}}{\sum_{i=1}^{r} \varpi_i(a(t))}$$

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(a(t))\{A_i(x(t))Z(x(t)) + B_i(x(t))u(t)\} \quad (2)$$

where

$$a(t) = [a_1(t) \ldots a_p(t)]$$

$$\varpi_i(a(t)) = \prod_{j=1}^{p} M_{ij}(a_j(t))$$

$$h_i(a(t)) = \frac{\varpi_i(a(t))}{\sum_{i=1}^{r} \varpi_i(a(t))}$$

$\varpi_i(a(t))$ is the matching degree (firing strenght) of the *i*-th rule and $h_i(a(t))$ is the normalized membership function.

3 Stable Polynomial Fuzzy Control Design

This section presents the design of polynomial fuzzy control systems based on a sum of squares approach.

3.1 Sum of Squares

In this paper, the computational method for polynomial fuzzy control design is based on the SOS decomposition of multivariate polynomials. A multivariate polynomial $f(x(t))$ where $x(t) \in R^n$ is a sum of squares (SOS) if there exist polynomials $f_1(x(t)),..., f_m(x(t))$ such that $f(x(t)) = \sum_{i=1}^{m} f_i^2(x(t))$. $f(x(t)) > 0$ for all $x(t) \in R^n$ [8].

The following lemma presents the connection between the SOS representation and the existence of apositive semidefinite matrix of the polynomial.

Lemma 1 [8]. Let $f(x(t))$ be a polynomial in $x(t) \in R^n$ of degree $2d$ and $Z(x(t))$ the column vector whose entries are all monomials in $x(t)$ with a degree no greater than *d*. Then $f(x(t))$ is SOS if there exists a positive semidefinite *P* such that

$$f(x(t)) = Z^T(x(t))PZ(x(t)) \tag{3}$$

Implication of the conditions for the positive semidefinite, SOS and existence of a positive semidefinite matrix of polynomials are presented by the following lemma.

Lemma 2 [9]. Let $f(x)$ be an $N \times N$ polynomial matrix of degree $2d$ in $x \in R^n$ and $Z(x)$ the column vector whose entries are all monomials with a degree no greater than *d*, and consider the following conditions.

(1) $F(x) \geq 0$ for all $x \in R^n$.

(2) $v^T F(x) v$ is SOS, where $v \in R^N$.

(3) There exists a positive semidefinite matrix Q such that $v^T F(x) v = (v \otimes Z(x))^T Q (v \otimes Z(x))$, where $\otimes$ denotes the *Kronecker product.*

Then, $(1) \Leftarrow (2)$ and $(2) \Leftrightarrow (3)$.

3.2 Polynomial Fuzzy Controller

A fuzzy controller with polynomial rule consequence [4] constructed from a polynomial fuzzy model (1) is represented as

Control rule i-th

$$\text{If } a_1(t) \text{ is } M_{i1} \text{ and ... and } a_p(t) \text{ is } M_{ip}$$

$$\text{Then } u(t) = -F_i(x(t))Z(x(t)),\ i = 1, \ldots, r. \tag{4}$$

The overall fuzzy controller is given by

$$u(t) = -\sum_{i=1}^{r} h_i(a(t))F_i(x(t))Z(x(t)). \tag{5}$$

From (2) and (5), the controlled system can be represented as

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(a(t))h_j(a(t)) \\ &= \left\{A_i(x(t)) - B_i(x(t))F_j(x(t))\right\} Z(x(t))\end{aligned} \tag{6}$$

3.3 Stable Controller Design

For stability analysis, a polynomial Lyapunov function represented by

$$Z^T(x(t))P(x(t))Z(x(t)) \tag{7}$$

is used, where $P(x(t))$ denotes a polynomial matrix in $x(t)$. If $Z(x(t)) = x(t)$ and $P(x(t))$ is a constant matrix, then (7) is reduced to the quadratic Lyapunov function $x(t)^T Px(t)$.

Lemma 3 presents the SOS conditions for the design of a stable polynomial fuzzy system.

Lemma 3 [4]. The control system consisting of (2) and (10) is stable if there exists a symmetric polynomial matrix $P(\tilde{x})R^{N\times N}$ and a polynomial matrix $M_i(x) \in R^{m\times N}$ such that (8) and (9) are fulfilled, where $\varepsilon_1(x) > 0$ for $x \neq 0$ and $\varepsilon_{2ij}(x) \geq 0$ for all x

$$v^T(P(\tilde{x}) - \varepsilon_1(x)I)v \text{ is SOS} \tag{8}$$

$$\begin{aligned}&-v^T(T(x)A_i(x)P(\tilde{x}) - T(x)B_i(x)M_j(x) + P(\tilde{x})A_i^T(x)T^T(x) - \\ &M_j^T(x)B_i^T(x)T^T(x) + T(x)A_j(x)P(\tilde{x}) - T(x)B_j(x)M_i(x) + \\ &P(\tilde{x})A_j^T(x)T^T(x) - M_i^T(x)B_j^T(x)T^T(x) -\end{aligned}$$

$$\sum_{k\in K}\frac{\partial P(\tilde{x})}{\partial x_k}A_i^{\,k}(x)Z(x)-\sum_{k\in K}\frac{\partial P(\tilde{x})}{\partial x_k}A_j^{\,k}(x)Z(x)+\varepsilon_{2ij}(x)I)v \text{ is SOS, } i<j, \quad (9)$$

where $v\in R^N$ is a vector independent of *x*. $T(x)\in R^{N\times n}$ is a polynomial matrix represented by

$$T^{ij}(x)=\frac{\partial Z_i}{\partial x_j}(x), \quad (10)$$

$A_i^k(x)$ signifies the *k*-th row of $A_i(x)$. $K=\{k_1,k_2,...,k_m\}$ signifies the row indices of $B_i(x)$ whose corresponding row is equal to zero, and $\tilde{x}$ is defined as $\tilde{x}=(x_{k1},x_{k2},...,x_{km})$.

Since $B_i^k(x)=0$ for $k\in K$, then

$$\dot{x}_k=\sum_{i=1}^{r}h_i(a)A_i^k(x)Z(x)$$

and for $i\notin K$,

$$\frac{\partial P(\tilde{x})}{\partial x_i}=0.$$

If (9) is satisfied, where $\varepsilon_{2ij}(x)>0$, then the zero equilibrium is asymptotically stable. If $P(\tilde{x})$ is a constant matrix, then the stability is globally satisfied.

The feedback gain $F_i(x)$ can be constructed from $P(\tilde{x})$ and $M_i(x)$ as

$$F_i(x)=M_i(x)P^{-1}(\tilde{x}). \quad (11)$$

4 Polynomial Fuzzy H_∞ Control Design

Nonlinear H_∞ control with an SOS approach has been proposed by Prajna, *et al*. in [9] for nonlinear dynamic plants. This method will be expanded in this paper for polynomial fuzzy control systems.

Consider the polynomial fuzzy model with disturbance and controller

$$\dot{x}=\sum_{i=1}^{r}h_i(a)\{A_i(x)Z(x)+B_{1i}(x)w+B_{2i}(x)u\} \quad (12)$$

$$u = -\sum_{i=1}^{r} h_i(a) F_i(x) Z(x) \tag{13}$$

and define

$$z = \sum_{i=1}^{r} h_i(a) \{ C_{1i}(x) Z(x) + u \} \tag{14}$$

where $C_{1i}(x)$ are also polynomial matrices. Vector w, u, and z denote the exogenous input, the control input, and the objective signal to be regulated, respectively. The objective is to design a state feedback law $u = -F_i(x)Z(x)$ such that the closed-loop system is asymptotically stable and L_2 gain from *w* to *z* is minimized.

Theorem 1. The zero equilibrium of the closed-loop polynomial fuzzy system (12) and (13) is asymptotically stable, and has an L_2 gain from *w* to *z* that is less than γ if there exist a symmetric polynomial matrix $P(\tilde{x}) \in R^{N \times N}$ and $M_i \in R^{n \times N}$ such that (15), (16), and (17) are satisfied, for SOS polynomials $\varepsilon_1(x) > 0$, $\varepsilon_{ii}(x) > 0$, and $\varepsilon_{ij}(x) > 0$ for $x \neq 0$.

$$v_1^T (P(\tilde{x}) - \varepsilon_1(x) I) v_1 \text{ is SOS} \tag{15}$$

$$-v_2^T \begin{bmatrix} S_{ii}(x) + \varepsilon_{ii}(x) I & T(x) B_{1i}(x) & P(\tilde{x}) C_{1i}^T(x) + M_j^T(x) \\ B_{1i}^T(x) T^T(x) & -\gamma^2 I & 0 \\ C_{1i}(x) P(\tilde{x}) + M_j(x) & 0 & I \end{bmatrix} v_2 \text{ is SOS for } i=j \tag{16}$$

$$-v_3^T \begin{bmatrix} \frac{1}{2}\left(S_{ij}(x) + S_{ji}(x)\right) + \varepsilon_{ij}(x) I & \frac{1}{2} T(x) \left(B_{1i}(x) + B_{1j}(x)\right) & \frac{1}{2} \begin{Bmatrix} \left(P(\tilde{x}) C_{1i}^T(x) - M_j^T(x)\right) + \\ \left(P(\tilde{x}) C_{1j}^T(x) - M_i^T(x)\right) \end{Bmatrix} \\ \frac{1}{2} T(x) \left(B_{1i}^T(x) + B_{1j}^T(x)\right) & -\gamma^2 I & 0 \\ \frac{1}{2} \begin{Bmatrix} \left(C_{1i}(x) P(\tilde{x}) - M_j(x)\right) + \\ \left(C_{1j}(x) P(\tilde{x}) - M_i(x)\right) \end{Bmatrix} & 0 & I \end{bmatrix} v_3$$

is SOS, for $i < j$ (17)

where

$$S_{ij}(x) = T(x) A_i(x) P(\tilde{x}) - T(x) B_{2i}(x) M_j(x) + P(\tilde{x}) A_i^T(x) T^T(x) - M_j^T(x) B_{2i}^T(x) T^T(x) - \left(\sum_{k=1}^{n} \frac{\partial P(\tilde{x})}{\partial x_k} A_i^k(x) Z(x) \right)$$

and $M_i(x) = F_i(x)P(\tilde{x})$.

Proof.

By using a state feedback controller, the state and output equations of the controlled system are represented by

$$\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(a)h_j(a)\{(A_i(x) - B_{2i}(x)F_j(x))Z(x) + B_{1i}(x)w\} \tag{18}$$

$$z = \sum_{i=1}^{r} h_i(a)\{(C_{1i}(x) - F_j(x))Z(x)\}. \tag{19}$$

If there exists a polynomial Lyapunov function $V(x) = Z^T(x)P^{-1}(\tilde{x})Z(x)$, $P(\tilde{x}) > 0$, and $\gamma \geq 0$, it follows that

$$\dot{V}(x) + z^T z - \gamma^2 w^T w \leq 0 \tag{20}$$

for the system described by (12) and (14). By assuming that initial condition $x(0) = 0$, integrating (20) from 0 to *T*, it follows that

$$V(x) + \int_0^T (z^T z - \gamma^2 w^T w)dt \leq 0. \tag{21}$$

Since $V(x) \geq 0$, it implies $\frac{\|z\|_2}{\|w\|_2} \leq \gamma$.

Therefore, the L_2 gain of the polynomial fuzzy system is less than γ if (20) is satisfied.

By substituting the Eqs. (18) and (19) into (20),we have

$$\dot{V}(x) + z^T z - \gamma^2 w^T w = Z^T(x)P^{-1}(\tilde{x})\dot{Z}(x) + \dot{Z}^T(x)P^{-1}(\tilde{x})Z(x) +$$
$$Z^T(x)\dot{P}^{-1}(\tilde{x})Z(x) + z^T z - \gamma^2 w^T w$$
$$\dot{V}(x) + z^T z - \gamma^2 w^T w =$$

$$\begin{bmatrix} Z(x) \\ w \end{bmatrix}^T \begin{bmatrix} \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(a)h_j(a)\{R_{ij}(x) + (C_{1i}(x) - F_j(x))^T (C_{1i}(x) - F_j(x))\} & \sum_{i=1}^{r} h_i(a)P^{-1}(\tilde{x})T(x)B_{1i}(x) \\ \sum_{i=1}^{r} h_i(a)B_{1i}^T(x)T^T(x)P^{-1}(\tilde{x}) & -\gamma^2 I \end{bmatrix} \begin{bmatrix} Z(x) \\ w \end{bmatrix} \leq 0, \tag{22}$$

where

$$R_{ij}(x) = P^{-1}(\tilde{x})T(x)\left(A_i(x) - B_{21}(x)F_j(x)\right) + \left(A_i(x) - B_{21}(x)F_j(x)\right)^T T^T(x)P^{-1}(\tilde{x}) + \left(\sum_{k=1}^{n} \frac{\partial P^{-1}(\tilde{x})}{\partial x_k} A_i^k(x)Z(x)\right).$$

From (22) we can obtain the following inequality

$$-\begin{bmatrix} \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(a)h_j(a)\left\{R_{ij}(x) + \left(C_{1i}(x) - F_j(x)\right)^T \left(C_{1i}(x) - F_j(x)\right)\right\} & \sum_{i=1}^{r} h_i(a)P^{-1}(\tilde{x})T(x)B_{1i}(x) \\ \sum_{i=1}^{r} h_i(a)B_{1i}^T(x)T^T(x)P^{-1}(\tilde{x}) & -\gamma^2 I \end{bmatrix} \geq 0, \quad (23)$$

for all x.

By multiplying both sides of (23) by blockdiag $\left[P(\tilde{x})\ I\right]$ and some algebraic manipulations, it follows that

$$-\begin{bmatrix} S_{ii}(x) & T(x)B_{1i}(x) & P(\tilde{x})C_{1i}^T(x) + M_j^T(x) \\ B_{1i}^T(x)T^T(x) & -\gamma^2 I & 0 \\ C_{1i}(x)P(\tilde{x}) + M_j(x) & 0 & I \end{bmatrix} \geq 0, \text{ for } i < j$$

and

$$-\begin{bmatrix} \frac{1}{2}\left(S_{ij}(x) + S_{ji}(x)\right) & \frac{1}{2}T(x)\left(B_{1i}(x) + B_{1j}(x)\right) & \frac{1}{2}\begin{Bmatrix} \left(P(\tilde{x})C_{1i}^T(x) - M_j^T(x)\right) + \\ \left(P(\tilde{x})C_{1j}^T(x) - M_i^T(x)\right) \end{Bmatrix} \\ \frac{1}{2}T(x)\left(B_{1i}^T(x) + B_{1j}^T(x)\right) & -\gamma^2 I & 0 \\ \frac{1}{2}\begin{Bmatrix} \left(C_{1i}(x)P(\tilde{x}) - M_j(x)\right) + \\ \left(C_{1j}(x)P(\tilde{x}) - M_i(x)\right) \end{Bmatrix} & 0 & I \end{bmatrix} \geq 0,$$

for $i < j$.

Using Lemma 2, the SOS conditions (16) and (17) imply that the matrices are positive semidefinite for all x.

5 Numerical Example

In this section a numerical example is presented to show the effectiveness of the proposed SOS method.

Consider a nonlinear system represented by

$$\dot{x} = A(x)x + B_1(x)w + B_2(x)u \tag{24}$$

$$z = C(x)x + u \,. \tag{25}$$

Here, $A(x) = \begin{bmatrix} -1 + x_1 - 1.5x_1^2 - 0.75x_2^2 & 0.25 - x_1^2 - 0.5x_2^2 \\ -\dfrac{\sin(x_1)}{x_1} & 0 \end{bmatrix}$

$$B_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{and } C(x) = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The nonlinear system (24) and (25) can be represented by a polynomial fuzzy model as follows :

Plant rule 1

$$IF\ \ a \text{ is } \mu_1(a), THEN \begin{cases} \dot{x} = A_1(x)x + B_{11}(x)w + B_{21}(x)u \\ z = C_1(x)x + u \end{cases}$$

Plant rule 2

$$IF\ \ a \text{ is } \mu_2(a), THEN \begin{cases} \dot{x} = A_1(x)x + B_{12}(x)w + B_{22}(x)u \\ z = C_2(x)x + u \end{cases}.$$

Here,

$$A_1(x) = \begin{bmatrix} -1 + x_1 - 1.5x_1^2 - 0.75x_2^2 & 0.25 - x_1^2 - 0.5x_2^2 \\ 0.2172 & 0 \end{bmatrix}$$

$$B_{11}(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_{21}(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1(x) = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$A_2(x) = \begin{bmatrix} -1 + x_1 - 1.5x_1^2 - 0.75x_2^2 & 0.25 - x_1^2 - 0.5x_2^2 \\ -1 & 0 \end{bmatrix}$$

$$B_{12}(x)=\begin{bmatrix}1\\0\end{bmatrix},\ B_{22}(x)=\begin{bmatrix}0\\1\end{bmatrix},\ C_2(x)=\begin{bmatrix}1 & 0\end{bmatrix}.$$

To reduce the computational complexity and to simplicity of the polynomial fuzzy system's representation, the non-linear part of the model system $a=-\frac{\sin x_1}{x_1}$ can be selected as fuzzy variable, where $a\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Since the maximum and minimum values of a are 0.2172 and -1, respectively, the membership functions $\mu_1(a)$ and $\mu_2(a)$ can be written as $\mu_1(a)=\frac{a+1}{1{,}2172}$ and $\mu_2(a)=\frac{0.2172-a}{1{,}2172}$, respectively. The membership functions $\mu_1(a)$ and $\mu_2(a)$ are obtained from the property of $\mu_1(a)+\mu_2(a)=1$. The membership functions $\mu_1(a)$ and $\mu_2(a)$ are depicted in Figure 2.

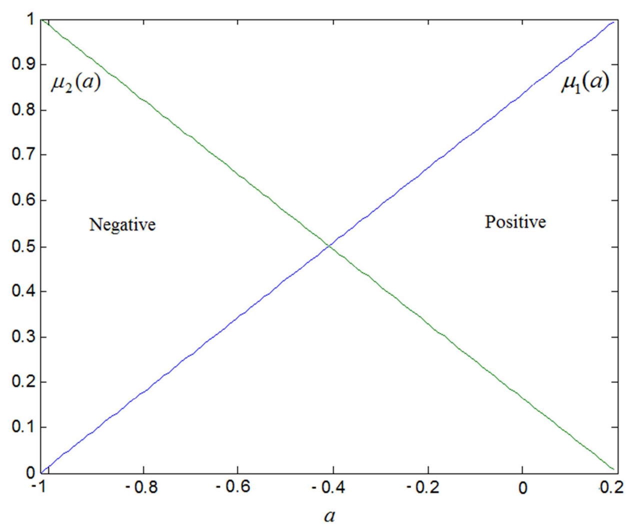

Figure 2 Membership functions $\mu_1(a)$ and $\mu_2(a)$.

An H_∞ control law for polynomial fuzzy systems is derived using Theorem 1 by minimizing γ . The values of $\varepsilon_1(x)$, $\varepsilon_{11}(x)$, $\varepsilon_{12}(x)$, and $\varepsilon_{22}(x)$ are chosen to be equal to the positive constant 0.001. In this example we design three controllers using matrix $P(\tilde{x})$ with various degrees. The solutions are obtained as

$P(\tilde{x})$ has degree zero

$$P(\tilde{x}) = \begin{bmatrix} 0.87238 & -0.43616 \\ -0.43616 & 0.65424 \end{bmatrix},\ M(x) = \begin{bmatrix} -0.86972 & 1.04 \\ 0.19211 & 0.82326 \end{bmatrix}, \text{with } \gamma = 1.14808,$$

$P(\tilde{x})$ has degree one

$$P(\tilde{x}) = \begin{bmatrix} 0.86366 + 0.04643x_1 & -0.38845 - 0.06685x_1 \\ -0.38845 - 0.06685x_1 & 0.75934 + 0.10267x_1 \end{bmatrix},$$

$$M(x) = \begin{bmatrix} -0.85963 & 871,87 \\ 0.21062 & 881,83 \end{bmatrix}, \text{with } \gamma = 1.14987,$$

$P(\tilde{x})$ has degree two

$$P(\tilde{x}) = \begin{bmatrix} 1.09206 - 1.27316x_1 + 1.66754x_1^2 & -0.4364 + 0.00112x_1 - 0.00137x_1^2 \\ -0.4364 + 0.00112x_1 - 0.00137x_1^2 & 0.6548 - 0.00262x_1 + 0.00207x_1^2 \end{bmatrix},$$

$$M(x) = \begin{bmatrix} -0.86962 & 244.62 \\ 0.19211 & 131.96 \end{bmatrix}, \text{with } \gamma = 1.14808.$$

The control input (13) will stabillize the closed-loop system, and the L_2 gain from w to z is no greater than 1.148, 1.150, and 1.148 for solutions of matrices $P(\tilde{x})$ and $M(x)$, which have order zero, one, and two, respectively.

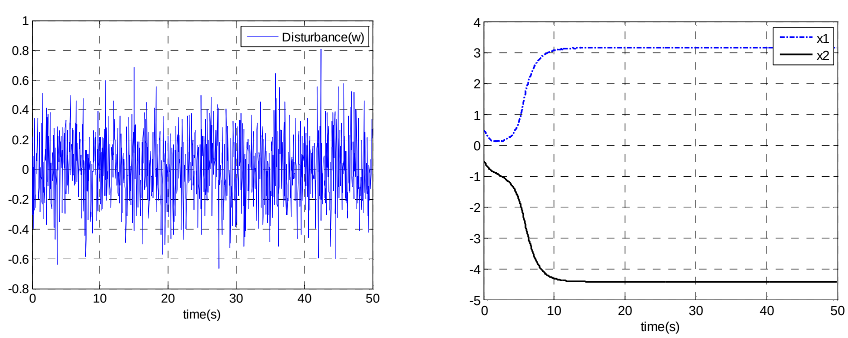

Figure 3 Disturbance (w) and state trajectories of the system without controller.

Figure 3 shows the time response behavior of the nonlinear system without controller for initial states x_1 and x_2 of 0.5 and of -0.5, respectively. It is found from the figure that states of the system x_1 and x_2 do not reach zero equilibrium; the state x_1 tends to 3.13994, and state x_2 tends to -4.42426. The random disturbance signal is applied to the system during simulation, which is also shown in Figure 3.

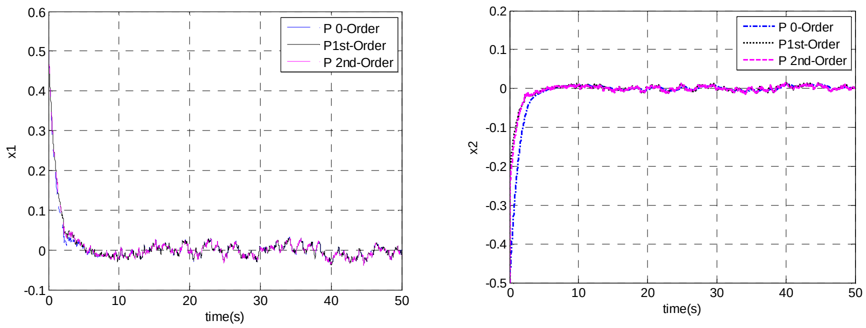

Figure 4 State trajectories of the closed-loop system with various degrees of the matrix $P(\tilde{x})$.

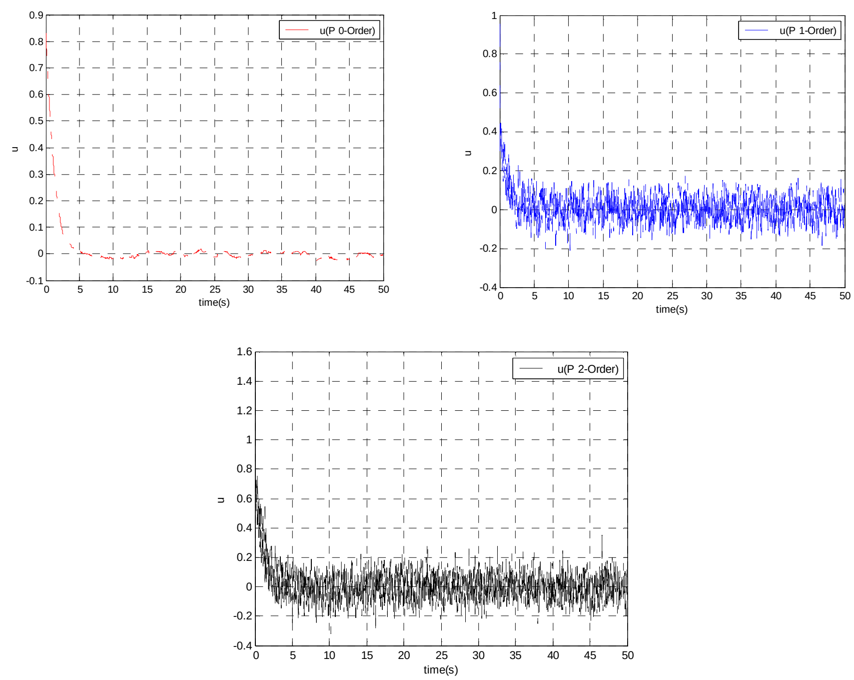

Figure 5 Input signals $u(t)$ with various degrees of matrix $P(\tilde{x})$.

The behavior of the closed-loop system for various degrees of matrix $P(\tilde{x})$ is shown in Figure 4. All of the close-loop system state trajectories tend to zero in spite of the given system disturbance. This reveals that the closed-loop system is stable. Furthermore, the disturbance input noise influence is reduced. During simulation, the initial state (x_1, x_2) was set at $(0.5,\text{-}0.5)$.

The input signal $u(t)$ is the output signal of the polynomial fuzzy controller applied to the plant. Figure 5 shows the control signal trajectory for various degrees of matrix $P(\tilde{x})$. As seen in Figure 5,the control signal is moving towards the region close to zero, which indicates that the polynomial fuzzy control signal stabilizes the closed-loop polynomial fuzzy system to the zero equilibrium.

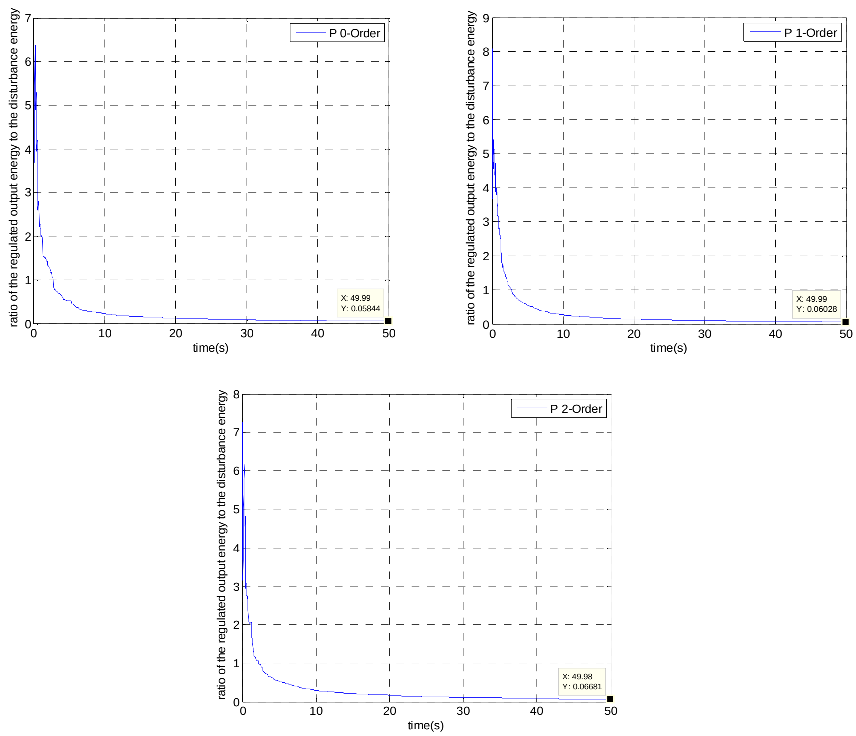

Figure 6 Ratio of the regulated output energy to the disturbance input energy.

The ratio of regulated output energy to disturbance input energy $\int_0^t z^T(x)z(x)dt \Big/ \int_0^t w^T(x)w(x)dt$ tends to constant values that are about 0.534, 0.559, and 0.553 for solutions of matrices $P(\tilde{x})$ and $M(x)$, which have order zero, one, and two, respectively. Therefore, the L_2 gain from w to z $\sqrt{\int_0^t z^T(x)z(x)dt \Big/ \int_0^t w^T(x)w(x)dt}$ is smaller than γ, which is obtained from the control design. The ratio of the regulated output energy to the disturbance input energy is shown in Figure 6.

The mean square error (MSE) performance criteria of the closed-loop system with the various degrees of matrix $P(\tilde{x})$ are shown in Figure 7. The MSE values of x_1 for degrees zero, one, and two of $P(\tilde{x})$ are 0.0042, 0.0047, and 0.0047, respectively. The MSE values of x_2 for degrees zero, one, and two of $P(\tilde{x})$ are 0.0053, 0.0010, and 0.0012, respectively.

By Lemma 2, the SOS conditions (17) and (18) imply that the matrices on the left hand side are positive semidefinite for all x, if condition 2 in this lemma is satisfied. Therefore semidefinite programming can be used to solve this problem.

Matrices $P(\tilde{x})$ and $M_i(x)$ can be found by searching the solution of (17) and (18). To find the solutions that meet these conditions, the SOSTOOLS program can be used [10]. The solution of the matrices is not unique. It was found that increasing the degree of solution matrix $P(\tilde{x})$ does not necessarily increase the performance of the closed-loop system.

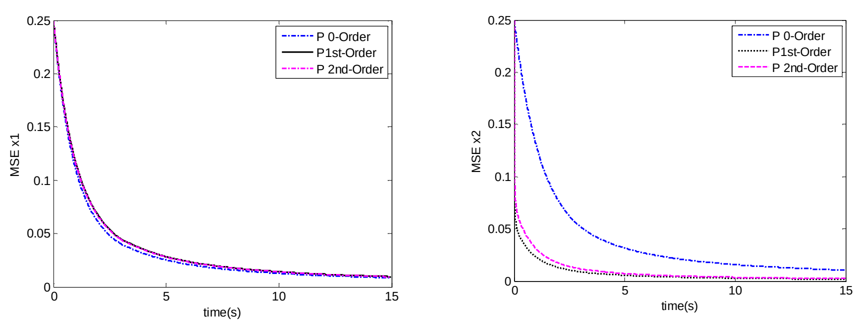

Figure 7 Mean square error of the closed-loop system states with various degrees of matrix $P(\tilde{x})$.

If $P(\tilde{x})$ and $M_i(x)$ are constant matrices and $Z(x) = x$, then the system representation reduces to the T-S fuzzy model and control, and the design conditions are represented as LMI.

The controller that satisfies the conditions such that the closed-loop system is asymptotically stable and minimizes L_2 gain from w to z can be constructed as $F_i(x) = M_i(x)P^{-1}(\tilde{x})$.

6 Conclusions

This paper has proposed H_∞ control of polynomial fuzzy systems using a sum of squares approach. The design condition is represented in terms of SOS. The solution of the condition can be solved with the SOSTOOLS program. The controller that satisfies the SOS condition ensures not only that the closed-loop system is asymptotically stable, but also that the L_2 gain from w to z is minimized. A numerical example has been presented to illustrate the validity of the H_∞ control design for polynomial fuzzy systems.

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