Interior topology: a new approach in topology

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ABSTRACT

This paper defined a new type of topology known as Interior topology. This work falls among the types of topology (such as general topology, supra topology, generalized topology, and filter) that are motivated by real-world concepts such as the orbits of planets around the sun, electron orbits around the nucleus, and so on. This form of topology is self-contained. The primary objective of this study is to respond to the question “Is general topology capable of producing Interior topology?”. Finally, we define the base for Interior topology which is called i-base.

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1. INTRODUCTION

Numerous researchers have introduced new topological structures via either the topological elements (open sets) or the topology itself (topology definition), such as Kelly [1] study of two topologies determine for the same set named Bitopology and introduced various separation properties into topological spaces, and obtained generalizations of some important classical results. Chang [2] introduced the definition of fuzzy topological spaces and extended straightforwardly some concepts of crisp topological spaces to fuzzy topological spaces. Then researchers [3], [4] introduced the concept of fuzzy Bitopological space and defined the compactness of fuzzy topological space and the continuity, closeness, and openness of mapping on the associated supra-fuzzy topological space. Shapir [5] defined soft topology on soft sets, and then Riza *et al.* [6], defined N-soft topology on N-soft sets, which is an extension of soft topology. Tarizadeh [7] defined flat topology in terms of the ring's prime spectrum. A flat topology is the dual of the Zariski topology [8], [9] thus Zariski topology of a ring R is a topology on the set of prime ideals, known as the ring spectrum. Its closed sets are $\upsilon(a)$, where a is any ideal in R and $\upsilon(a)$ is the set of prime ideals containing a, and many others who introduced new topological structures.

These structures were influenced by an idea, a relationship, the outside world, or natural phenomena. They may or may not be real, but they will lay the framework for establishing the correct scientific underpinnings, if we look up into the sky, a lot of thoughts come into our mind, and one of these thoughts is "Can we divide the vast universe into portions under particular conditions (topology)?". The purpose of this is to gain a deeper understanding and to broaden our perceptions. Here, we will discuss mathematical ideas, as some may believe it is simply a discipline concerned with the language of numbers, calculating, and symbols. Indeed, we cannot blame people for this belief, but it is a reality that everyone should be aware of mathematics is the language of the sciences, and it is the most precise approach for proving hypotheses and claims across a wide variety of fields.

Nonetheless, we are attempting in our effort to introduce a novel idea of topology, which we have dubbed (Interior topology). The structure of this topology corresponds to some phenomena around us, such as the phenomenon of the movement of planets around the sun and the phenomenon of the movement of electrons around the nucleus. This structure must be clearly defined and based on known topological foundations. We will build a collection in which the infinite intersection is nonempty and attempt to demonstrate its existence using examples and comparisons to various topologies such as general topology, supra topology [10], generalized topology [11] and so filter [12].

The structure of Interior topology differs from that of other types of topologies as general topology and supra topology. We can study their topological properties to get a better understanding of the things around us. Finally, we shall have open sets known as i-open and so closed sets known as i-closed.

2. PRELIMINARIES

Definition 1: Let X be a set. A topology on X is a subclass $T \subseteq P(X)$ of subsets of X, called open sets (shr. Open), such that satisfied the following:

a. $\emptyset$ and X are open.

b. The intersection of finitely many open sets is open.

c. Any union (finite or infinite) of open set is open.

A topological space is set X together with topology T on X.

Definition 2 [10]: Let X be a set. A subclass $T^* \subseteq P(X)$ is called supra topology on X if:

a. $X \in T^*$.

b. T^* is closed under an arbitrary union of elements of T^*.

A supra topological space is set X together with supra topology T^* on X

Remark 1: Every topological is supra topological space but the converse is not necessary as follows:

Example 1: Let X be any infinite set and T^* be a collection of all subsets which have more than one point, then its obvious that T^* is supra topology but it isn't topology on X.

Definition 3 [11]-[13]: Let X be a set. A subclass $\mu \subseteq P(X)$ is called generalized topology on X if:

a. $\emptyset \in \mu$.

b. μ is closed under an arbitrary union of elements of μ.

A generalized topological space is set X together with generalized topology μ on X.

Remark 2: it's clear that every topological is generalized topological space but the converse is not necessary.

Remark 3: There exists no relation between generalized topological and supra-topological space.

Definition 4 [14]: Let X be a nonempty set. A subclass $\mathcal{F} \subseteq P(X)$ is called a filter on X if the following is satisfied:

a. $\emptyset \notin \mathcal{F}$.

b. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

c. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$ then $B \in \mathcal{F}$.

Remark 4: There exists no relation between filter and topology.

3. INTERIOR TOPOLOGY

Definition 5: Let X be a nonempty set. A subclass $I_t \subseteq P(X)$ is called Interior topology on X if the following is satisfied:

a. $\emptyset \notin I_t$.

b. I_t is closed under an arbitrary union of elements of I_t.

c. I_t is closed under the arbitrary intersection of elements of I_t.

An Interior topological space is set X together with the Interior topology I_t on X.

Example 2: Let $X = \{a, b, c\}$ and $I_t = \{\{a, b\}, \{b, c\}, \{b\}, X\}$ then I_t is satisfy the conditions of Interior topology, thus (X, I_t) is interior topological space.

Remark 5: Maybe X does not belong to I_t as follow:

Example 3: Let $X = \{a, b, c, d\}$ and $I_t = \{\{a, b\}, \{b, c\}, \{b\}, \{a, b, c\}\}$ then I_t is Interior topology on X but $X \notin I_t$, thus (X, I_t) is interior topological space.
Note 1: If we want to image the structure of definition 5, we can represent it as follows:
These orbits (Figure 1) suggest to us the shape of the orbits around the sun, and the region S originated from the intersections of all these orbits. Now, we will discover the types of sets in this space.

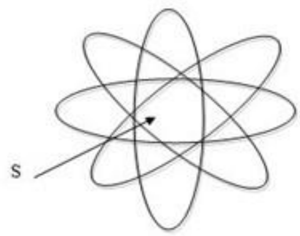

Figure 1. Orbits

Definition 6: Let (X, I_t) be an interior topological space, then the element $U \in I_t$ is called i-open set (shr. i-open) and the complement of U is well called i-closed set (shr. i-closed), therefore we will reformulate definition 5 as follow:
Definition 7: Let X be a nonempty set and I_t be a collection of subsets of X (which is named i-open) then I_t is an Interior topology on X if satisfy the following:
a. $\emptyset$ is not i-open.
b. The union of i-open sets is i-open set.
c. The intersection of i-open sets is i-open set.

Accordingly, we note the i-closed sets will satisfy dual of the above conditions as follow:
Proposition 1: Let (X, I_t) be an interior topological space then the set of all i-closed satisfy the following:
a*. X is not i-closed.
b*. The union of i- closed sets is i- closed set.
c*. The intersection of i- closed sets is i- closed set.
Proof:
a* For each i-open set $U \in I_t$, U^c is i-closed, therefore $\bigcap_{j\in J} U_j$ for each $j \in J$ is not exist in any i-closed set, thus X is impossible is i-closed set.
b* & c* is explain from complement from b & c.
Note 2: From above, we note the i-open set is impossible is i-closed set because it contains the intersection of all i-open sets which is not contained in any i-closed set.
Definition 8: Let (X, I_t) be an interior topological space and let $A \subseteq X$, $l \in A$ is called i-limit point of A if $l \in U, \forall U \in I_t$. The set of i-limit points of A is called i-limit set of A. The set of all i-limit points of all subsets in X is called the target set.

The target set satisfies some properties as follows:
Proposition 2: Let (X, I_t) be an interior topological space then the following properties are equivalent:
a. $E \subseteq X$ is the target set.
b. $E = \bigcap_{j\in J} U_j$, for each $j \in J$.
c. E is a minimal i-open set.
Proof:
$a \Rightarrow b$
Let E is the target set in X, $<$To prove $E = \bigcap_{j\in J} U_j$, for each $j \in J >$
Let $l \in E$, then from definition 4, $l \in \bigcap_{j\in J} U_j \ \forall U_j \in I_t \Rightarrow E \subseteq \bigcap_{j\in J} U_j$, $\forall U_j \in I_t$ and at the same time every element $l \in \bigcap_{j\in J} U_j$, for each $j \in J$ is i-limit point of set $\bigcap_{j\in J} U_j \Rightarrow l \in E \Rightarrow \bigcap_{j\in J} U_j \subseteq E$, thus $E = \bigcap_{j\in J} U_j$.
$b \Rightarrow c$
If $E = \bigcap_{j\in J} U_j$, $\forall U_j \in I_t$, then E is contained in any i-open set therefore, E is a minimal i-open set.
$c \Rightarrow a$
Let μ be the minimal i-open set and E is the target set in X then $\mu \subseteq E$ it's clear.
If $E \nsubseteq \mu \Rightarrow \exists l \in E$ such that $l \notin \mu$, l is i-limit point for some subset of X and l is contained in all i-open sets $\Rightarrow$ $l \in \bigcap_{j\in J} U_j$ for each $j \in J \ \Rightarrow \bigcap_{j\in J} U_j$ is not minimal i-open $\Rightarrow$ contradiction, therefore, $E \subseteq \mu \Rightarrow \mu = E$.

Example 4: In real line $\mathbb{R}$, let $I_t = \{S_n = \left(0,1+\frac{1}{n}\right), n \in N\} \cup [0,1]$ then I_t are Interior topology on $\mathbb{R}$ and the target set is $[0,1]$.
Example 5: In real line $\mathbb{R}$, let $I_t = \{S_{n,m} = [1-\frac{1}{n}, 2+\frac{1}{m}], n, m \in N\}$ then I_t are Interior topology on $\mathbb{R}$ and the target set is $[1,2]$.
Example 6: In real line $\mathbb{R}$, let $I_t = \{S_n = (0+n, 1+n), n \in N\}$ then I_t are Interior topology on $\mathbb{R}$ and the target set is $[1,2]$.
Proposition 3: In any interior topological space, the target set is existing and unique.
Proof: Let (X, I_t) be an interior topological space, since $\bigcap_{i\in I} U_i$, for each U_i in I_t, is i-open set therefore, the target set exists.
Let E_1, E_2 are two target sets in I_t then $E_1 = \bigcap_{i\in I} U_i = E_2$, for each U_i in I_t, therefore the target set is unique.
Note 1: In Figure 1, we may imagine the region S as the target set.

3.1. Interior relative topology

Proposition 4: Let (X, I_t) be an interior topological space and $S \subseteq X$ is the target set in I_t and let $G \subseteq X$ such that $G \cap S \neq \emptyset$ then the following collection $I_{t_G} = \{U \cap G: \forall U \in I_t\}$ is an Interior topology on G.
Proof: Let $V_j \in I_{t_G}, j \in J$, then $V_j = U_j \cap G$, thus

a. $\bigcup_{j\in J} V_j = \bigcup_{j\in J}(U_j \cap G) = \bigcup_{j\in J} U_j \cap G \in I_{t_G}$.
b. $\bigcap_{j\in J} V_j = \bigcap_{j\in J}(U_j \cap G) = \bigcap_{j\in J} U_j \cap G \in I_{t_G}$.
c. If $\emptyset \in I_{t_G} \Rightarrow \exists H \in I_t$ such that $\emptyset = H \cap G \Rightarrow (S \cap H) \cap (S \cap G) = S \cap (S \cap G) = S \cap G = \emptyset$ and this is a contradiction, therefore $\emptyset \notin I_{t_G}$.

Now, we will define the interior relative topology.

Definition 9: The collection above I_{t_G} is called interior relative topology on subset $G \subseteq X$ in Interior topology I_t and (G, I_{t_G}) is called interior relative topological space.
Remark 6: Note if $G \cap S = \emptyset$, G, and S as defined in proposition 4, then I_{t_G} is not satisfied the conditions of Interior topology because $\emptyset \in I_{t_G}$, therefore the relative topology is defined only on a subset $G \subseteq X$ which intersection with the target set and this is one of the reasons why it is called Interior topology.

4. THE RELATION BETWEEN INTERIOR TOPOLOGY AND OTHER TOPOLOGIES

Here, we will prove there is no relation between Interior topology and topology, generalized topology, supra topology, and filter. This section aims to prove the independence and existence of Interior topology.

4.1. Topology and interior topology

Every topology is impossible Interior topology and versa vice; we note that from Definitions 1 and 5, thus the topology has $\emptyset$ but the Interior topology hasn't $\emptyset$, in addition, the condition of intersections mostly does not come true in topology. But there is a relation between them we will discuss later in section 5.

4.2. Supra topology and interior topology

The Supra topology maybe is Interior topology or not, i.e: i) Supra topology is Interior topology as example 2; ii) Supra topology isn't Interior topology as remark 1. such that if we have supra topology which is also topology then it is not topology as sub-section 4.1; and iii) Interior topology is not supra topology as example 3.

4.3. The generalized topology and interior topology

As sub-section 4.1, the generalized topology is impossible in Interior topology because it has $\emptyset$ and versa vice.

4.4. The filter and Interior topology

The filter maybe Interior topology or not as follows:

a. The filter isn't Interior topology as in the following example:

Example 7: In the real usual topological space, let F be a collection of neighborhoods of number 0, then F is filtered on 0 but it isn't Interior topology since the infinite intersection doesn't belong to F.

b. The filter is Interior topology as in example 2, the collection I_t is satisfied with the conditions of the filter and at the same time, it is Interior topology.
c. Interior topology isn't filtered as the following example:

Example 8: In example 4, I_t is Interior topology but it isn't filtered since $(0,1+\frac{1}{3}) \in I_t$ but $(0,1+\frac{1}{3}] \notin I_t$ such that condition 3 of definition 4 isn't satisfied.

Now, we will discuss the necessary condition which makes the filter Interior topology.

Proposition 5: Let (X,T) be a topological space and let $F \neq \emptyset$ filter in X, if F has the nonempty infinite intersection of all it elements then F is Interior topology.

Proof: Let $F \neq \emptyset$ be a filter in a topological space (X,T) and let $S \neq \emptyset$ is the infinite intersection of all elements of F:

a. Let $A_\ell \in F, \ell \in J$, $S \in F$ but $S \subseteq \bigcap_{\ell\in J} A_\ell$ therefore $\bigcap_{\ell\in J} A_\ell \in F$ by condition 3 of definition 4.

b. Let $B_\ell \in F, \ell \in J$, but $B_\ell \subseteq \bigcup_{\ell\in J} B_\ell$ therefore $\bigcup_{\ell\in J} B_\ell \in F$ by condition 3 of definition 4.

c. $\emptyset \notin F$ by condition 1 of definition 4.

From above then F is the Interior topology on X.

Remark 7: If we add the set S, which is meaning the infinite intersection of all elements of F, to filter F then F can become not filter as the following example:

Example 9: In example 7, if we add $\{0\}$, which is the infinite intersection of all elements of F, to F then F becomes not filter because the condition 3 of definition 4 such that for instance $\{0\} \subseteq \{0,1,2\} \Rightarrow \{0,1,2\} \in F$ but $\{0,1,2\}$ isn't neighborhood of 0.

5. I-BASE OF INTERIOR TOPOLOGY

Definition 10: Let (X, I_t) be an interior topological space. The collection $\beta_t \subseteq I_t$ is called I-base for Interior topology I_t if every i-open is the intersection of elements of β_t.

Example 10: In example 2, the I-base for I_t is $\beta_t = \{\{a,b\},\{b,c\},X\}$.

Example 11: In example 4, the I-base for I_t is $\beta_t = \{S_n = \left(0,1+\frac{1}{n}\right), n \in N\}$.

Now, we will be discussing the following question: Is the topology generated by Interior topology?

Theorem 1: Let (X,T) be a topological space and $\emptyset \neq A \subseteq X$. Let $\beta_t \subseteq T$ be collection contains A and all elements of T which contains A then β_t is I-base for Interior topology I_t on target set A.

Proof: Let (X,T) be a topological space and $\emptyset \neq A \subseteq X$. Let $\beta_t \subseteq T$ be collection contains A and all elements of T which contains A. Now, we suppose I_t is a collection containing all intersections of elements of β_t then:

a. Let $\{V_j\}_{j\in J} \in I_t$, $V_j = \bigcap_{\ell\in L} B^j_\ell, B^j_\ell \in \beta_t\ \forall j \in J$ then $\bigcap_{j\in J} V_j = \bigcap_{j\in J}\{\bigcap_{\ell\in L} B^j_\ell\} = \bigcap_{j\in J}\bigcap_{\ell\in L} B^j_\ell$ and this means $\bigcap_{j\in J} V_j$ is the intersection of elements of β_t therefore $\bigcap_{j\in J} V_j \in I_t$.

b. Let $\{U_j\}_{j\in J} \in I_t$, $U_j = \bigcap_{\ell\in L} C^j_\ell, C^j_\ell \in \beta_t\ \forall j \in J$ then $\bigcup_{j\in J} U_j = \bigcup_{j\in J}\{\bigcap_{\ell\in L} C^j_\ell\} = \bigcap_{\ell\in L}\bigcup_{j\in J} C^j_\ell$ (by [15]), thus $\bigcup_{j\in J} C^j_\ell$ are elements in T that have A, therefore, belong to β_t, thus $\bigcap_{j\in J} V_j$ is the intersection of elements of β_t therefore $\bigcap_{j\in J} V_j \in I_t$.

c. From definition 4 and since the target set $A \neq \emptyset$, then $\emptyset \notin I_t$.

6. CONCLUSION

We defined an Interior topology in this work as one that is formed using an i-open set. However, we investigated the structure's independence from the previously indicated structures. I-base also talks about how the relationship between Interior topology and general topology is shown.

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