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Research Paper

Enumerate the Number of Vertices Labeled Connected Graph of Order Seven Containing No Parallel Edges

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Abstract

A graph that is connected *G*(V,E) is a graph in which there is at least one path connecting every two vertices in G; otherwise, it is called a disconnected graph. Labels or values can be assigned to the vertices or edges of a graph. A vertex-labeled graph is one in which only the vertices are labeled, and an edges-labeled graph is one in which only edges are assigned values or labels. If both vertices and edges are labeled, the graph is referred to as total labeling. If given *n* vertices and m edges, numerous graphs can be made, either connected or disconnected. This study will be discussed the number of disconnected vertices labeled graphs of order seven containing no parallel edges and may contain loops. The results show that number of vertices labeled connected graph of order seven with no parallel edges is $N(G_{7,m,g})_l$= 6,727×C_6^m; while for 7≤g≤ 21, $N(G_{7,m,g})_l$= k_g $C_{g-1}^{(m-(g-6))}$, where k_7 =30,160, k_8 = 30,765, k_9 =21,000, k_{10} =28,364, k_{11}= 26,880, k_{12} =26,460 , k_{13} = 20,790, k_{14} =10,290, k_{15} = 8,022, k_{16} = 2,940, k_{17} =4,417, k_{18} = 2,835, k_{19} = 210, k_{20} = 21, k_{21} = 1.

Keywords

Vertex Labeled, Connected Graph, Order Seven, Loops

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1. INTRODUCTION

Without any doubt, one of the most widely used fields of mathematics is graph theory, especially to represent a real-life problem because of the flexibility of drawing a graph. The date of birth of graph theory began with the publication of Euler's solution regarding the Konigsberg problem in 1736, one of the mathematical fields with specific birth date is graph theory (Vasudev, 2006). Given a set V={v_1, v_2, $\cdots$, v_n} of vertices/nodes, V≠∅, and a set of edges E={e_{ij}|i,j∈V}, a graph *G*(V,E) is a structure that consists of ordered pair of V and E. In real-life problems, the cities, buildings, airports, and others can be portrayed by vertices, while the roads that connect the cities, the pipes that connect the buildings, the flight paths that connect the airports, and others can be portrayed by edges. In order to represent real-life problems, on the edges, we can assign nonstructural information such as distance, time, flow, cost, and others by setting a non-negative number c_{ij}≥0.

There is a lot of application of graph theory in real-life problems, such as in computer science, chemistry, biology, sociology, agriculture, and others. In computer science, the graph structure plays an important role, for example, in designing a database, software engineering, and network system (Singh, 2014). The database is used for interconnecting analysis, as a storage system with index-free adjacency, as a tool for graph-like-query, and for other purposes. In network design, graph-based representation makes the problem easier to visualize and provides a more accurate definition. In agriculture, the dynamic closures of the accounting structure are represented by a directed graph (Álvarez and Ehnts, 2015). The structure of graphs, together with discrete mathematics, are applied in chemistry to model the biological and physical properties of chemical compounds (Burch, 2019). The theoretical graph concept also was used by Gramatica et al. (2014) to represent or describe the possible modes of action of a given pharmacological compound. In biology, a phylogenetic tree was represented by a leaf-labeled tree (Huson and Bryant, 2006; Brandes and Cornelsen, 2009), while Mathur and Adlakha (2016) represented DNA using a combined tree. Hsu and Lin (2008) presented many graph theoretical concepts in engineering and computer science, and Al Etaiwi (2014) used the concepts of a complete graph, cycle graph, and minimum spanning tree to generate a complex cipher text. Priyadarsini (2015) explored the use of graph theory concepts, expander, and extremal graphs, in the design of some ciphers, whereas Ni et al. (2021) created ciphers using corona and bipartite graphs. In agriculture, graph theory concepts were used to group agricultural workers performing manual tasks (Kawakura and Shibasaki, 2018), while the concept of graph coloring to optimize a farmer's goal was used by Kannimuthu et al. (2020). The relationship and unification of graph theory and physical-chemical measures (such as boiling and melting point, covalent and ionic potentials, and electronic density) make molecular topology can describe molecular structure comprehensively. A weighted directed graph, connectivity matrix, and Dijkstra's algorithm were used by Holmes et al. (2021) in plasma chemical reaction engineering. The basic structure of a directed graph is mostly used for the visualization of the reactions. Moreover, they use Gelphi, an open-source graph software for visualization.

In 1874, Cayley counted the number of hydrocarbon isomers C_nH_{2n+2} (Cayley, 1874), and this process is similar to enumerating the number of a binary tree. Bona (2007) discussed the method of enumerating trees and forests. Redfield and Pólya are two other mathematicians that worked independently with graphical enumeration, especially in graph coloring (Bogart, 2004), and in graph enumeration, a comprehensive explanation of Pólya's counting theorem is one of the most powerful tools.

The number of graphs that can be formed for labeled and unlabeled graphs is different if we are given n vertices and m edges. For example, given n= 3 and m= 2, the number of simple connected unlabeled graphs that can be constructed is only one, while if every vertex is assigned labeled, The maximum number of graphs that can be created is three. The higher the order of the graph, the more labeled graph are formed. Agnarsson and Greenlaw (2006) gave the formula to enumerate graphs. However, no formula for enumerating graphs with special properties such as planarity or connectivity was provided.

There are some studies that have been done concerning the enumeration of the vertex-labeled graph with connectivity properties. In 2017, Amanto et al. (2017) proposed the formula to count disconnected vertices labeled graphs of order maximal four. For order five, the number of labeled vertices in connected graphs with no loops and may contain maximal five parallel edges had been proposed by Wamiliana et al. (2019). Amanto et al. (2021) studied the relationship between the formula for the number of connected vertex labels with no loops in graphs of order five and order six. Wamiliana et al. (2020) also discussed the number of vertices labeled connected graphs of order six with no parallel edges and a maximum of ten loops, while Puri et al. (2021) gave the formula to compute the number of vertices labeled connected graphs of order six without loops, while Ansori et al. (2021) proposed the number of vertices labeled connected graphs of order seven with no loops.

The article is organized as follows: Section I provides information about graphs, graph applications in various fields, and previous research related to this topic. Section II discusses Observation and Investigation, while Section III discusses Results and Discussion. Section IV contains the conclusion.

2. OBSERVATION AND INVESTIGATION

Suppose that we are given the number of vertices n= 7, and the number of edges m. We will construct connected graphs G(V,E) of order n. Since the graph must be connected, then $m \geq 6$. Moreover, every vertex is labeled. Let g as the number of non-loop edges, $g \geq n-1$.

We start firstly by constructing all basic patterns of connected graphs of order seven. Note that the basic patterns contain no loops. The basic pattern starts with m= 6, and with n= 7, m= 6, and constructs all possible patterns. After all possible patterns for m= 6 are already constructed, then we continue with m= 7, and so on until m= 21. When m= 21, only one pattern can be constructed because parallel edges are not allowed. Figure 1 shows some examples of patterns for m= 6, Figure 2 shows some patterns that are isomorphic with the first graph in the second row of the graphs in Figure 1, and Figure 3 shows when m= 21.

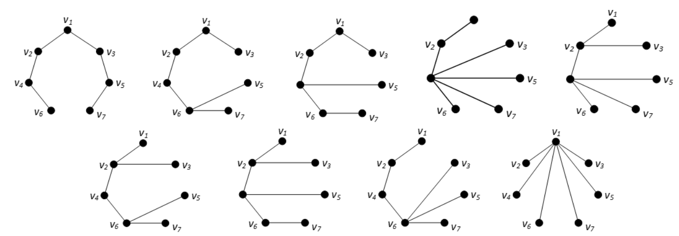


Figure 1. Some Basic Patterns for n= 7 and m= 6

Note that all isomorphic graphs will be counted in the pattern. However, we do not need to construct isomorphic graphs. Figure 2 shows the patterns of isomorphic graphs of the pattern of the first graph in the second row of Figure 1.

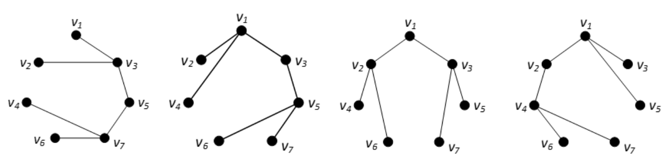


Figure 2. Some Patterns are Isomorphic with the First Graph in the Second Row in Figure 1

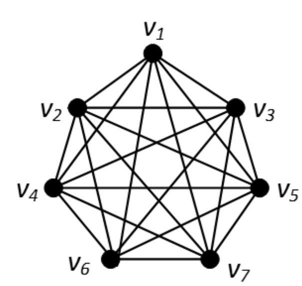


Figure 3. The Basic Pattern for n= 7 and m= 21

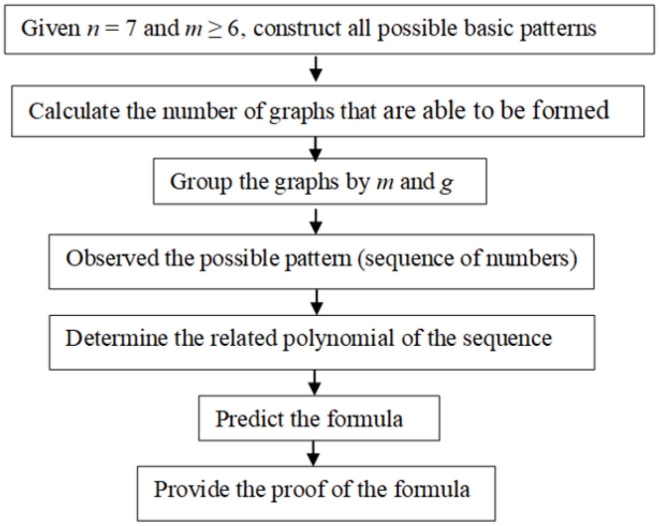


Figure 4. The Procedure

After constructing the basic pattern, the enumeration step begins. It begins from the first pattern of n= 7 dan m= 6 by adding one loop so that m= 7, calculating the number of graphs that are able to be formed, and then continuing with this pattern by increasing the number of loops (m= 9), and so on. Continue with this similar manner until the last pattern. The procedure can be put in the following diagram:

3. RESULTS AND DISCUSSION

The first step, as given in Figure 4 is constructing all possible patterns. Because there are many patterns obtained and due to limitation of space, here we give some patterns and also the number of all possible graphs formed according to the patterns. The obtained graphs are grouped by m and g, for example, for n=6, m= 6, and g= 6, the patterns are:
The results for all patterns are shown in Table 1 below:

Please note that in the table the dash sign (–) means there is impossible to construct the graph, while the empty space on the table means that we are not calculate more because g is fixed in each column, adding more edges simply adds more loops, and the constructed graph already constitute a sequence of numbers. The number in each column is able to be written as multiplication of a fix number and a sequence of number so that Table 2 can be rewritten in Table 3 as follow:

From Table 3 we can see that for every g= 6, 7, $\cdots$, 21, the number of graphs obtained are bigger as m increases, and the number of graphs obtained are multiplication of a fix number. For example, for g= 6, the fix number is 6,727, and the number of graphs increases follows a certain pattens of sequence which is 1, 7, 28, 84, 210, 462, 924, 1,716, 3,003, 5,005.

1 7 28 84 210 462 924 1716 3003 5005
6 21 56 126 252 462 792 1287 2002
15 35 70 126 210 330 495 715

Table 1. The Pattern for n= 7, m= 6, and g= 6

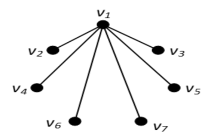
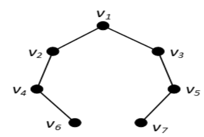
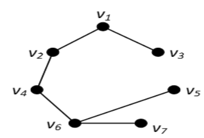
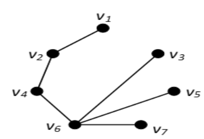
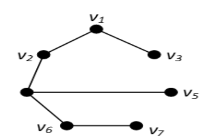
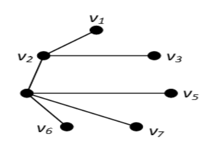
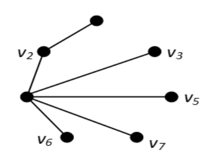
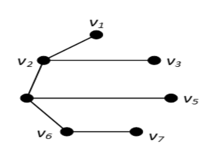
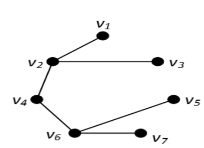
The patterns	The number of isomorphic graphs
	$C^7_1.C^6_6$= 7
	$\frac{7!}{2}$= 2,520
	$C^7_1.C^6_3.C^3_2$= 420
	$C^7_1.C^6_2.C^4_3.1$= 420
	$C^7_1.C^6_3.3!$= 840
	$C^7_1.C^6_1.C^5_2.C^3_3$= 420
	$C^7_1.C^6_1.C^5_1.C^4_4$= 210
	$C^7_2.C^5_1.C^4_2.2!$= 1,260
	$C^7_2.C^5_1.C^4_2.C^2_2$= 630
Total	6,727

Table 2. The Number of Vertices Labeled Connected Graph of Order Seven Containing No Parallel Edges

m	g = 6	7	8	9	10	11
	The number of vertices labeled connected order seven graphs with no parallel edges					
6	6,727	–	–	–	–	–
7	47,089	30,160	–	–	–	–
8	188,356	211,120	30,765	–	–	–
9	565,068	844,480	215,355	21,000	–	–
10	1,412,670	2,533,440	861,420	147,000	28,364	–
11	3,107,874	6,333,600	2,584,260	588,000	198,548	26,880
12	6,215,748	13,933,920	6,460,650	1,764,000	794,192	188,160
13	11,543,532	27,867,840	14,213,430	4,410,000	2,382,576	752,640
14	20,201,181	51,754,560	28,426,860	9,702,000	5,956,440	2,257,920
15	33,668,635	90,570,480	52,792,740	19,404,000	13,104,168	5,644,800
16	–	150,950,800	92,387,295	36,036,000	26,208,336	12,418,560
17	–	–	153,978,825	63,063,000	48,672,624	24,837,120
18	–	–	–	105,105,000	85,177,092	46,126,080
19	–	–	–	–	141,961,820	80,720,640
20	–	–	–	–	–	134,534,400

m	g = 12	13	14	15	16
	The number of vertices labeled connected order seven graphs with no parallel edges				
12	26,460	–	–	–	–
13	185,220	20,790	–	–	–
14	740,880	145,530	10,290	–	–
15	2,222,640	582,120	72,030	8,022	–
16	5,556,600	1,746,360	288,120	56,154	2,940
17	12,224,520	4,365,900	864,360	224,616	20,580
18	24,449,040	9,604,980	2,160,900	673,848	82,320
19	45,405,360	19,209,960	4,753,980	1,684,620	246,960
20	79,459,380	35,675,640	9,507,960	3,706,164	617,400
21	132,432,300	62,432,370	17,657,640	7,412,328	1,358,280
22	–	104,053,950	30,900,870	13,765,752	2,716,560
23	–	–	51,501,450	24,090,066	5,045,040
24	–	–	–	40,150,110	8,828,820
25	–	–	–	–	14,714,700

m	g = 12	13	14	15	16
	The number of vertices labeled connected order seven graphs with no parallel edges				
17	4,417	–	–	–	–
18	30,919	2,835	–	–	–
19	123,676	19,845	210	–	–
20	371,028	79,380	1,470	21	–
21	927,570	238,140	5,880	147	1
22	2,040,654	595,350	17,640	588	7
23	4,081,308	1,309,770	44,100	1,764	28
24	7,579,572	2,619,540	97,020	4,410	84
25	13,264,251	4,864,860	194,040	97,02	210
26	22,107,085	8,513,505	360,360	19,404	462
27	–	14,189,175	630,630	36,036	924
28	–	–	1,051,050	63,063	1,716
29	–	–	–	105,105	3,003
30	–	–	–	–	5,005

Table 3. Alternative form of Table 2

m	The number of vertices labeled connected order seven graphs with no parallel edges					
	g					
	6	7	8	9	10	11
6	1x6,727	–	–	–	–	–
7	7x6,727	1x30,160	–	–	–	–
8	28x6,727	7x30,160	1x30,765	–	–	–
9	84x6,727	28x30,160	7x30,765	1x21,000	–	–
10	210x6,727	84x30,160	28x30,765	7x21,000	1x28,364	–
11	462x6,727	210x30,160	84x30,765	28x21,000	7x28,364	1x26,880
12	924x6,727	462x30,160	210x30,765	84x21,000	28x28,364	7x26,880
13	1,716x6,727	924x30,160	462x30,765	210x21,000	84x28,364	28x26,880
14	3,003x6,727	1,716x30,160	924x30,765	462x21,000	210x28,364	84x26,880
15	5,005x6,727	3,003x30,160	1,716x30,765	924x21,000	462x28,364	210x26,880
16	–	5,005x30,160	3,003x30,765	1,716x21,000	924x28,364	462x26,880
17	–	–	5,005x30,765	3,003x21,000	1,716x28,364	924x26,880
18	–	–	–	5,005x21,000	3,003x28,364	1,716x26,880
19	–	–	–	–	5,005x28,364	3,003x26,880
20	–	–	–	–	–	5,005x26,880

m	The number of vertices labeled connected order seven graphs with no parallel edges				
	g				
	12	13	14	15	16
12	1x26,460	–	–	–	–
13	7x26,460	1x20,790	–	–	–
14	28x26,460	7x20,790	1x10,290	–	–
15	84x26,460	28x20,790	7x10,290	1x8022	–
16	210x26,460	84x20,790	18x10,290	7x8,022	1x2,940
17	462x26,460	210x20,790	84x10,290	28x8,022	7x2,940
18	924x26,460	462x20,790	210x10,290	84x8,022	28x2,940
19	1,716x26,460	924x20,790	462x10,290	210x8,022	84x2,940
20	3,003x26,460	1,716x20,790	924x10,290	462x8,022	210x2,940
21	5,005x26,460	3,003x20,790	1,716x10,290	924x8,022	462x2,940
22	–	5,005x20,790	3,003x10,290	1,716x8,022	924x2,940
23	–	–	5,005x10,290	3,003x8,022	1,716x2,940
24	–	–	–	5,005x8,022	3,003x2,940
25	–	–	–	–	5,005x2,940

m	The number of vertices labeled connected order seven graphs with no parallel edges				
	g				
	12	13	14	15	16
17	1x4,417	–	–	–	–
18	7x4,417	1x2,835	–	–	–
19	28x4,417	7x2,835	1x210	–	–
20	84x4,417	28x2,835	7x210	1x21	–
21	210x4,417	84x2,835	28x210	7x21	1x1
22	462x4,417	210x2,835	84x210	28x21	7x1
23	924x4,417	462x2,835	210x210	84x21	28x1
24	1,716x4,417	924x2,835	462x210	210x21	84x1
25	3,003x4,417	1,716x2,835	924x210	462x21	210x1
26	5,005x4,417	3,003x2,835	1,716x210	924x21	462x1
27	–	5,005x2,835	3,003x210	1,716x21	924x1
28	–	–	5,005x210	3,003x21	1,716x1
29	–	–	–	5,005x21	3,003x1
30	–	–	–	–	5,005x1

20 35 56 84 120 165 220
15 21 28 36 45 55
6 7 8 9 10
1 1 1 1

Notate $N(G_{7,m,g})_l$ as the number of vertices labeled connected graphs of order seven containing no parallel edges (loops are allowable) with the number of edges is m and the number of non loop edges is g.

Result 1: Given $m \geq 6$, $g = 6$, the total number of vertices labeled connected graphs of order seven with no parallel edges is $N(G_{7,m,g})_l = 6{,}727 \times C_6^m$

Proof:

Consider the above sequence of numbers.

That sequence of numbers is able to be represented by polynomial of order six because the fixed

$$Q_5 m = \alpha_6 m^6 + \alpha_5 m^5 + \alpha_4 m^4 + \alpha_3 m^3 + \alpha_2 m^2 + \alpha_1 m + \alpha_0$$

The following system of equations is obtained by substituting m= 6, 7, 8, 9, 10, 11, 12 to $Q_5(m)$.

$$6,727 = 46,656\alpha_6 + 7,776\alpha_5 + 1,296\alpha_4 + 216\alpha_3 + 36\alpha_2 + 6\alpha_1 + \alpha_0 \quad (1)$$

$$47,089 = 117,649\alpha_6 + 16,807\alpha_5 + 2,401\alpha_4 + 343\alpha_3 + 49\alpha_2 + 7\alpha_1 + \alpha_0 \quad (2)$$

$$188,356 = 262,144\alpha_6 + 32,768\alpha_5 + 4,096\alpha_4 + 512\alpha_3 + 64\alpha_2 + 8\alpha_1 + \alpha_0 \quad (3)$$

$$565,068 = 531,441\alpha_6 + 59,049\alpha_5 + 6,561\alpha_4 + 729\alpha_3 + 81\alpha_2 + 9\alpha_1 + \alpha_0 \quad (4)$$

$$1,412,670 = 1,000,000\alpha_6 + 100,000\alpha_5 + 10,000\alpha_4 + 1,000\alpha_3 + 100\alpha_2 + 10\alpha_1 + \alpha_0 \quad (5)$$

$$3,107,874 = 1,771,561\alpha_6 + 161,051\alpha_5 + 14,641\alpha_4 + 1,331\alpha_3 + 121\alpha_2 + 11\alpha_1 + \alpha_0 \quad (6)$$

$$6,215,748 = 2,985,984\alpha_6 + 248,832\alpha_5 + 20,736\alpha_4 + 1,728\alpha_3 + 144\alpha_2 + 12\alpha_1 + \alpha_0 \quad (7)$$

These equations form a system of equations that can be transformed into a matrix $Ax = b$ as follow:

$$\begin{bmatrix} 46,656 & 7,776 & 1,296 & 216 & 36 & 6 & 1 \\ 117,649 & 16,807 & 2,401 & 343 & 49 & 7 & 1 \\ 262,144 & 32,768 & 4,096 & 512 & 64 & 8 & 1 \\ 531,441 & 59,049 & 6,561 & 729 & 81 & 9 & 1 \\ 1,000,000 & 100,000 & 10,000 & 1,000 & 100 & 10 & 1 \\ 1,771,561 & 161,051 & 14,641 & 1,331 & 121 & 11 & 1 \\ 2,985,984 & 248,832 & 20,736 & 1,728 & 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} \alpha_6 \\ \alpha_5 \\ \alpha_4 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 6,727 \\ 47,089 \\ 188,356 \\ 565,068 \\ 1,412,670 \\ 3,107,874 \\ 6,215,748 \end{bmatrix}$$

By solving this system of equations we get: $\alpha_6 = \frac{6{,}727}{720}$, $\alpha_5 = -\frac{100{,}905}{720}$, $\alpha_4 = \frac{57{,}175}{720}$, $\alpha_3 = -\frac{151{,}575}{720}$, $\alpha_2 = \frac{1{,}843{,}198}{720}$, $\alpha_1 = -\frac{807{,}240}{720}$, $\alpha_0 = 0$.

Thus

$$\begin{aligned} Q_5(m) &= \alpha_6 m^6 + \alpha_5 m^5 + \alpha_4 m^4 + \alpha_3 m^3 + \alpha_2 m^2 + \alpha_1 m^+ \alpha_0 \\ &= \frac{6,727}{720} m^6 - \frac{100,905}{720} m^5 + \frac{57,175}{720} m^4 - \frac{151,575}{720} m^3 \\ &\quad + \frac{1,843,198}{720} m^2 - \frac{807,240}{720} m \\ &= \frac{6,727}{720} (m^6 - 15m^5 + 85m^4 - 225m^3 + 274m^2 - 120m) \\ &= \frac{6,727(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)!}{6.5.4.3.2.1(m-6)!} \\ &= 6,727 \times C_6^m \end{aligned}$$

Please note that for every g (every column, the sequence of numbers is the same, except the multiplier). Thus, the polynomial related to the sequence of numbers is the same. However, because the multipliers are different, it will cause different formulas.

Result 2: Given $m \geq 7$, $g = 7$, the total number of vertices labeled connected graphs of order seven with no parallel edges is $N(G_{7,m,g})_l = 30{,}160 \times C_6^{(m-1)}$

Proof:

The polynomial that represents the sequence is the same which is

$$Q_5 m = \alpha_6 m^6 + \alpha_5 m^5 + \alpha_4 m^4 + \alpha_3 m^3 + \alpha_2 m^2 + \alpha_1 m + \alpha_0$$

However, for m= 7, the numbers of graphs are different. The following system of equations is obtained by substituting m= 7, 8, 9, 10, 11, 12, 13 to the equation.

$$30,160 = 117,649\alpha_6 + 16,807\alpha_5 + 2,401\alpha_4 + 343\alpha_3 + 49\alpha_2 + 7\alpha_1 + \alpha_0 \quad (8)$$

$$211,120 = 262,144\alpha_6 + 32,768\alpha_5 + 4,096\alpha_4 + 512\alpha_3 + 64\alpha_2 + 8\alpha_1 + \alpha_0 \quad (9)$$

$$844,480 = 531,441\alpha_6 + 59,049\alpha_5 + 6,561\alpha_4 + 729\alpha_3 + 81\alpha_2 + 9\alpha_1 + \alpha_0 \quad (10)$$

$$2,533,440 = 1,000,000\alpha_6 + 100,000\alpha_5 + 10,000\alpha_4 + 1,000\alpha_3 + 100\alpha_2 + 10\alpha_1 + \alpha_0 \quad (11)$$

$$6,333,600 = 1,771,561\alpha_6 + 161,051\alpha_5 + 14,641\alpha_4 + 1,331\alpha_3 + 121\alpha_2 + 11\alpha_1 + \alpha_0 \quad (12)$$

$$13,933,920 = 2,985,984\alpha_6 + 248,832\alpha_5 + 20,736\alpha_4 + 1,728\alpha_3 + 144\alpha_2 + 12\alpha_1 + \alpha_0 \quad (13)$$

$$27,867,840 = 4,826,809\alpha_6 + 371,293\alpha_5 + 28,561\alpha_4 + 2,197\alpha_3 + 169\alpha_2 + 13\alpha_1 + \alpha_0 \quad (14)$$

These equations form a system of equations that can be transformed into a matrix $Ax = b$ as follow:

$$\begin{bmatrix} 117,649 & 16,80 & 2401 & 343 & 49 & 7 & 1 \\ 262,144 & 32,768 & 4096 & 512 & 64 & 8 & 1 \\ 531,441 & 59,049 & 6561 & 729 & 81 & 9 & 1 \\ 1,000,000 & 100,000 & 10,000 & 1000 & 100 & 10 & 1 \\ 1,771,561 & 161,051 & 14,641 & 1331 & 121 & 11 & 1 \\ 2,985,984 & 248,832 & 20,736 & 1728 & 144 & 12 & 1 \\ 4,826,809 & 371,293 & 28,561 & 2197 & 169 & 13 & 1 \end{bmatrix} \begin{bmatrix} \alpha_6 \\ \alpha_5 \\ \alpha_4 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 30,160 \\ 211,120 \\ 844,480 \\ 2,533,440 \\ 6,333,600 \\ 13,933,920 \\ 27,867,840 \end{bmatrix}$$

By solving this system of equations we get: $\alpha_6 = \frac{30{,}160}{720}$, $\alpha_5 = -\frac{633{,}360}{720}$, $\alpha_4 = \frac{5{,}278{,}000}{720}$, $\alpha_3 = -\frac{22{,}167{,}600}{720}$, $\alpha_2 = \frac{48{,}979{,}840}{720}$, $\alpha_1 = -\frac{53{,}202{,}240}{720}$, and $\alpha_0 = \frac{21{,}715{,}200}{720}$.

Thus

$$\begin{aligned} Q_5(m) &= \alpha_6 m^6 + \alpha_5 m^5 + \alpha_4 m^4 + \alpha_3 m^3 + \alpha_2 m^2 + \alpha_1 m^+ \alpha_0 \\ &= \frac{30,160}{720} m^6 - \frac{633,360}{720} m^5 + \frac{5,278,000}{720} m^4 - \frac{22,167,600}{720} m^3 \\ &\quad + \frac{48,979,840}{720} m^2 - \frac{53,202,240}{720} m + \frac{21,715,200}{720} \\ &= \frac{30,160}{720} (m^6 - 21m^5 + 175m^4 - 735m^3 + 624m^2 - 1764m + 720) \\ &= \frac{30,160}{720} (m-1)(m-2)(m-3)(m-4)(m-5)(m-6) \\ &= \frac{30,160(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)(m-7)!}{6.5.4.3.2.1(m-7)!} \\ &= 30,160 \times C_6^{(m-1)} \end{aligned}$$

The following results are obtained by doing the similar manner:

For $m \geq 8$, g=8, is $N(G_{7,m,g})_l = 30{,}765 \times C_7^{(m-2)}$

For $m \geq 9$, g=9, is $N(G_{7,m,g})_l = 21{,}000 \times C_8^{(m-3)}$

For $m \geq 10$, g=10, is $N(G_{7,m,g})_l = 28{,}364 \times C_9^{(m-4)}$

For $m \geq 11$, g=11, is $N(G_{7,m,g})_l = 26{,}880 \times C_{10}^{(m-5)}$

For $m \geq 12$, g=12, is $N(G_{7,m,g})_l = 26{,}460 \times C_{11}^{(m-6)}$

For $m \geq 13$, g=13, is $N(G_{7,m,g})_l = 20{,}790 \times C_{12}^{(m-7)}$

For $m \geq 14$, g=14, is $N(G_{7,m,g})_l = 10{,}290 \times C_{13}^{(m-8)}$

For $m \geq 15$, g=15, is $N(G_{7,m,g})_l$ = 8,022×$C_{14}^{(m-9)}$
For $m \geq 16$, g=16, is $N(G_{7,m,g})_l$ = 2,940×$C_{15}^{(m-10)}$
For $m \geq 17$, g=17, is $N(G_{7,m,g})_l$ = 4,417×$C_{16}^{(m-11)}$
For $m \geq 18$, g=18, is $N(G_{7,m,g})_l$ = 2,835×$C_{17}^{(m-12)}$
For $m \geq 19$, g=19, is $N(G_{7,m,g})_l$ = 210×$C_{18}^{(m-13)}$
For $m \geq 20$, g=20, is $N(G_{7,m,g})_l$ = 21×$C_{19}^{(m-14)}$
For $m \geq 21$, g=21, is $N(G_{7,m,g})_l$ = $C_{20}^{(m-15)}$

Note that the multiplier for g= 6 is the same as the multiplier of t= 6 in Ansori et al. (2021), as well as g=7 with t= 7, and so on until g=21 with t=21. However, the formulas are different because in Ansori et al. (2021) the formula are for connected vertex labeled graph without loops while in this study is for connected vertices labeled graph without parallel edges. For example, for g=8, $N(G_{7,m,g})_l$ = 30,765×$C_7^{(m-2)}$, while in Ansori et al. (2021), for t= 8, $N(G_{7,m,8})$ = 30,765×$C_7^{(m-1)}$.

4. CONCLUSIONS

Based on the above reasoning, we may conclude that the number of vertices in a labeled connected graph of order seven with no parallel edges is $N(G_{7,m,g})_l$= 6,727×C_6^m for g= 6, while for $7 \leq g \leq 21$, $N(G_{7,m,g})_l$= k_g $C_{g-1}^{(m-(g-6))}$, where k_7 =30,160, k_8 = 30,765, k_9 =21,000, k_{10} =28,364, k_{11}= 26,880, k_{12} =26,460 , k_{13} = 20,790, k_{14} =10,290, k_{15} = 8,022, k_{16} = 2,940, k_{17} =4,417, k_{18} = 2,835, k_{19} = 210, k_{20} = 21, k_{21} = 1.

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