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Some Properties of Coprime Graph of Dihedral Group D_{2n} When n is a prime power

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Abstract. Research on algebra structures represented graph theory led the way to a new topic of research in recent years. In this paper, the algebraic structure that will be represented in the coprime graph is the dihedral group and its subgroup. The coprime graph of a group G, denoted by Γ_G is a graph whose vertices are elements of G and two distinct vertices a and b are adjacent if only if $(|a|, |b|) = 1$. Some properties of coprime graphs from a dihedral group D_{2n} are obtained. Some of the results is if n is prime then $\Gamma_{D_{2n}}$ is complete bipartite graph and if n is composite then $\Gamma_{D_{2n}}$ is multipartite graph.
Keywords: complate bipartite graph, coprime graph,dihedral group,multipartite graph.

I. INTRODUCTION

Study of algebraic structures represented in a graphs raises many recent and in interesting results. This area is relatively new, and over the years different types of graphs of a group were defined. For example, prime graph, the non-commuting graph, and Cayley graphs, which have a long history. In 2014, Ma et al define a coprime graph of a group [1]. The coprime graph of a group G, denoted by Γ_G is a graph whose vertices are elements of G and two distinct vertices a and b are adjacent if only if $(|a|, |b|) = 1$ [2]. Dorbidi in 2016 classify all the groups which Γ_G is a complete t-partite graph or a planar graph, he also studied the automorphism group of Γ_G [3]. In this paper, we will study some properties of coprime graph of a dihedral group.

II. RESULT

Some definitions of a group and a graph that used in this paper are given as follows.

Definition 1 ([4]) *A dihedral group G with identity e, is a group generated by two elements a, b with properties*

$$G = \{a, b | a^n = e, b^2 = e, bab^{-1} = a^{-1}\}$$

It is easy to see that the order of dihedral group is $2n$ and we denote dihedral with order $2n$ by D_{2n}. We can also rewrite $D_{2n} = \{e, a, a_2, a_3, \cdots a^{n-1}, b, ab; a^2b, a^3b, \cdots, a^{n-1}\}$. For example, for $n = 4$ we have $D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3\}$.

Definition 2 *If G is a group with identity e and $x \in G$, the order of x is the least natural number k such that $x^k = e$ and we write $|x| = k$.*

Definition 3 ([1]) *The coprime graph of a group G, denoted by Γ_G is a graph whose vertices are elements of G and two distinct vertices u and v are adjacent if and only if $(|a|, |b|) = 1$.*

2.1. Coprime Graph of D_{2n}

Given dihedral group D_{2n}, we will give some properties of $\Gamma_{D_{2n}}$ and Γ_S for all non trivial subgroup S of D_{2n}. The result is quite interesting, the coprime graph of D_{2n} always a tripartite graph except for n is a power of two.

First result, if n is prime then the coprime graph of D_{2n} is given by the following Theorem.

Theorem 1 *Let If n is an odd prime number, then the coprime graph $\Gamma_{D_{2n}}$ is a complete tripartite graph.*

Proof. Let n is an odd prime number, Define three partition by $V_1 = \{e\}$, $V_2 = \{a, a^2, \cdots, a^{p-1}\}$ and $V_3 = \{b, ab, a^2b, \cdot\cdot\cdot, a^{p-1}b\}$. Clearly $|e| = 1$, as n is prime number. Since the order of a is n and the order of b is 2 then we have $|a| = |a^2| = \cdots = |a^{n-1}| = n$, and $|b| = |ab| = |a^2b| = \cdots = |a^{n-1}b| = 2$. Hence for each $x, y \in V_i$ then $(|x|, |y|) = p \neq 1$ or $(|x|, |y|) = 2 \neq 1$, $i \in \{2, 3\}$. Then for any $u \in V_i$ dan $v \in V_j$ where $i \neq j$ we have $(|u|, |v|) = 1$, thus u and v are adjacent so the coprime graph of the dihedral group is complete tripartite graph. □

Second case is, if n is the power of 2, then the coprime graph of D_{2n} is given by the following Theorem.

Theorem 2 *Let $n = 2^k$, for some $k \in N$ then the coprime graph $\Gamma_{D_{2n}}$ is a complete bipartite graph.*

Proof. Let $D_{2n} = \{e, a, a^2, a^3, \cdots, a^{2^k-1}, b, ab, a^2b, a^3b, \cdots, a^{2^k-1}b\}$ is a dihedral group. Define two partitions $V_1 = \{e\}$, and $V_2 = \{a, a^2, \cdots, a^{2^k-1}, b, ab, a^2b, a^3b, \cdots, a^{2^k-1}b\}$. Clearly $|e| = 1$, as $n = 2^k$ then we have $|a| = |a^2| = \cdots = |a^{2^k-1}| = 2^s$ and $|b| = |ab| = |a^2b| = \cdots = |2^k - 1b| = 2$. Hence for each $x, y \in V_2$ we have $(|x|, |y|) = 2^s \neq 1$ for some $s \in \mathbb{N}$, thus x and y are not neighbors. Since $|e| = 1$ and for any $v \in V_2$ we have $(|e|, |v|) = 1$. We have e is adjacent to $v \in V_2$, then the Coprime graph of the dihedral group is a complete bipartite graph. □

Third case is, if n is prime power, then the coprime graph of D_{2n} is given by the following Theorem.

Theorem 3 *Let $n = p^k, p \neq 2$ and p is prime number, for some $k \in N$ then the coprime graph $\Gamma_{D_{2n}}$ is a complete tripartite graph.*

Proof. Let $D_{2n} = \{e, a, a^2, \cdot\cdot\cdot, a^{p^k-1}, b, ab, a^2b, \cdot\cdot\cdot, a^{p^k-1}b\}$ is a dihedral group. Define three partitions $V_1 = \{e\}$, $V_2 = \{e, a, a^2, \cdot\cdot\cdot, a^{p^k-1}\}$ and $V_3 = \{b, ab, a^2b, \cdot\cdot\cdot, a^{p^k-1}b\}$. Clearly $|e| = 1$, since $n = p^k, p \neq 2$ then $|a|, |a^2|, \cdot\cdot\cdot, |a^{p^k-1}|$ is divided by p and $|b| = |ab| = |a^2b| = \cdots = |a^{p^k-1}b| = 2$. Hence for each $x, y \in V_2$ we have $(|x|, |y|)$ is power of p or $(|x|, |y|) = 2 \neq 1$, then for any $u \in V_i$ and $v \in V_j$ where $i \neq j$ then $(|u|, |v|) = 1$. Hence u and v are adjacent,and the coprime graph of the dihedral group is a complete tripartite graph. □

2.2. Coprime Graph of Nontrivial Subgroup of D_{2n}

In this section we give the coprime graph of a subgroup of a dihedral group. The result is quite interesting, the coprime graph of nontrivial subgroups of D_{2n} is always a complete bipartite graph except for n is a power of odd prime. If n is a power of odd prime then there is a chance that the coprime graph of subgroup D_{2n} is a complete tripartite graph.

First, if n is prime then coprime graph of any subgroup of D_{2n} is given by the following Theorem.

Theorem 4 *Let S is a nontrivial subgroup of D_{2n}. If n is prime number, then the coprime graph Γ_S is a complete bipartite graph.*

Proof. From [5] we know all the subgroup of D_{2n} can be categorized into several subgroups, so we will divided into 2 cases

Case 1 (Rotation subgroup)

The subgroup contain identities and all rotation element, write $S_1 = \{e, a, a^2, \cdots, a^{n-1}\}$. Define two partitions $V_1 = \{e\}$, $V_2 = \{e, a, a^2, \cdots, a^{n-1}\}$. Clearly $|e| = 1$, as n is prime number and $|a| = |a^2| = \cdots = |a^{p-1}| = n$. Hence for each $x, y \in V_2$ then $(|x|, |y|) = n \neq 1$, then x and y are not neighbors. Since $|e| = 1$, then for any $v \in V_2$ we have $(|e|, |v|) = 1$. Then the coprime graph of subgroup S_1 is complete bipartite graph.

Case 2 (Reflection subgroup)

The subgroups contains identity and one reflection element, they are $S_0 = \{e, b\}, S_1 = \{e, ab\}, \cdots, S_{n-1} = \{e, a^{n-1}b\}$. Let $i \in \{0, ..., n-1\}$. Define two partitions $V_1 = \{e\}$, and $V_2 = \{a^ib\}$. Since $|e| = 1$ and $|a^ib| = 2$ then for any $v \in V_2$ we have $(|e|, |v|) = 1$. So e is adjacent to any $v \in V_2$. Then the coprime graph of the subgroups S_i is complete bipartite graph. □

Second, if n is the power of 2, then coprime graph of any subgroup of D_{2n} is given by the following Theorem.

Theorem 5 *Let S is a nontrivial subgroup of D_{2n}. If $n = 2^k$, for some $k \in N$ then the coprime graph of Γ_S is a complete bipartite graph.*

Proof. From [5] we know all the subgroup of D_{2n} can be categorized into several subgroups, so we will divided into 3 cases

Case 1 (Rotation subgroup)

The subgroup contain identities and some rotation element, they are $S_0 = \{e, a, a^2, \cdots, a^{2^k-1}\}$ the subgroup with all rotation element and $S_i = \{e, a^{2^i}, a^2 \cdots 2^{2i}, \cdots, a^{n-2^i}\}$ for $i \in \{2, \cdots, k-1\}$ Define two partitions $V_1 = \{e\}$ and $V_2 = S_i - \{e\}$. Clearly $|e| = 1$, and since $n = 2^k$ then $|a|, |a^2|, \cdots, |a^{2^k-1}|$ is divided by 2. Hence for each $x, y \in V_2$ then $(|x|, |y|)$ is power of 2, then x and y are not neighbors. Clearly for any $v \in V_2$ we have $(|e|, |v|) = 1$, so e is adjacent to any $v \in V_2$. Hence the coprime graph of the subgroup S_i is complete bipartite graph.

Case 2 (Reflection subgroup)

The subgroups contains identity and one reflection element, they are $S_1 = \{e, b\}, S_2 = \{e, ab\}, \cdots, S_n = \{e, a^{2^k-1}b\}$. Define two partitions $V_1 = \{e\}$, and $V_2 = \{a^i b\}$, for some $0 \leq i \leq n-1$. Since $|e| = 1$ and $|a^i b| = 2$ then for any $v \in V_2$ we have $(|e|, |v|) = 1$. Hence the coprime graph of the subgroups S_i is complete bipartite graph.

Case 3 (Mixed subgroup)

The subgroups containing identity with some reflection element and some rotation element,they are such as $S_i = \{e, a^{2^i}, a^{2\cdot 2^i}, \cdots, a^{n-2^i}b, a^{j+2^i}b, \cdots, a^{n-2^i+j}b\}$. Define two partitions $V_1 = \{e\}$, and $V_2 = S_i - \{e\}$. Clearly $|e| = 1$, $|a^{2^i}| = \cdots = |a^{n-2^i}| = 2^s$, for some $s \in \mathbb{N}$ and $s \leq k$, and $|b| = |a^{2^i}b| = \cdots = a^{n-2^i}b| = 2$. Hence for each $x, y \in V_2$ we have $(|x|, |y|) = 2 \neq 1$, then x and y are not neighbors. Since $|e| = 1$, for any $v \in V_2$ we have $(|e|, |v|) = 1$. Then e is adjacent to any $v \in V_2$. Hence the coprime graph of the subgroup S_i is a complete bipartite graph. □

Third, if n is a prime power, then coprime graph of any subgroup of D_{2n} is given by the following Theorem.

Theorem 6 *Let S is a nontrivial subgroup of D_{2n}. If $n = p^k$, p is prime number, for some $k \in N$ then the coprime graph Γ_S is a complete bipartite graph or a complete tripartite graph.*

Proof. From [5] we know all the subgroup of D_{2n} can be categorized into several subgroups, so we will divided into 3 cases

Case 1 (Rotation subgroup)

The subgroup contain identities and some rotation element, they are $S_0 = \{e, a, \cdots, a^{p^k-1}\}$ the subgroup with all rotation element and $S_i = \{e, a^{p^i}, a^2 \cdots p^{2i}, \cdots, a^{n-p^i}\}$ for $i \in \{2, \cdots, k-1\}$ Define two partitions $V_1 = \{e\}$ and $V_2 = S_i - \{e\}$. Clearly $|e| = 1$, and since $n = p^k$ then $|a|, |a^2|, \cdots, |a^{2^k-1}|$ is divided by p. Hence for each $x, y \in V_2$ then $(|x|, |y|)$ is the power of p, then x and y are not neighbors. Clearly for any $v \in V_2$ we have $(|e|, |v|) = 1$, so e is adjacent to any $v \in V_2$. Hence the coprime graph of the subgroup S_i is complete bipartite graph.

Case 2 (Reflection subgroup)

The subgroups contains identity and one reflection element, they are $S_1 = \{e, b\}, S_2 = \{e, ab\}, \cdots, S_n = \{e, a^{p^k-1}b\}$. Define two partitions $V_1 = \{e\}$, and $V_2 = \{a^i b\}$, for some $0 \leq i \leq n-1$. Since $|e| = 1$ and $|a^i b| = 2$ then for any $v \in V_2$ we have $(|e|, |v|) = 1$. Hence the coprime graph of the subgroups S_i is complete bipartite graph.

Case 3 (Mixed subgroup)

The subgroups containing identity with some reflection element and some rotation element,they are such as $S_i = \{e, a^{p^i}, a^{2\cdot p^i}, \cdots, a^{n-p^i}b, a^{j+p^i}b, \cdots, a^{n-p^i+j}b\}$. Define three partitions $V_1 = \{e\}$, $V_2 = \{a^{2^i}, a^{2\cdot 2^i}, \cdots, a^{n-2^i}\}$,and $V_3 = \{b, a^{p^i}b, \cdots, a^{n-p^i}b\}$. Clearly $|e| = 1$, since $n = p^k$ then $|a^{p^i}| = \cdots = a^{n-p^i} = p^s$, for some $s \in \mathbb{N}$ and $s \leq k$. And $|b| = |a^{p^i}b| = \cdots = a^{n-p^i}b| = 2$. Hence for each $x, y \in V_i$ we have $(|x|, |y|) = p^s \neq 1$ or $(|x|, |y|) = 2 \neq 1$ for $i \in \{2, 3\}$. For any $u \in V_i$ and $v \in V_j$ where $i \neq j$ then $(|u|, |v|) = 1$. So, u and v are adjacent. Hence the coprime graph of subgroup S_i is a complete tripartite graph. □

The last result valid for every subgroup of any dihedral group, even for any group.

Theorem 7 *If the coprime graph of the dihedral group is a complete bipartite graph, then the coprime graph of the nontrivial subgroups is a complete bipartite graph.*

Proof. Suppose the coprime graph of the dihedral group is in the form of a complete bipartite graph then $\exists V_1, V_2 \subset D_{2n}$ so that for each $v_1 \in V_1$ and $v_2 \in V_2$,$(|v_1|, |v_2|) = 1$ and for every $a, b \in V_i$, $i = 1, 2, (|a|, |b|) \neq 1$. Let $H < D_{2n}$, choose $W_1 = V_1 \bigcap H$ and $W_2 = V_2 \bigcap H$, hence if $x \in W_1$ then $x \in V_1$ and $y \in W_2$ thus $y \in_2$ as a result $(|x|, |y|) = 1$. So, the coprime graph of the subgroups is a complete bipartite graph. $\square$

III. CONCLUSIONS AND FUTURE RESEARCH DIRECTION

The obtained result show that the coprime graph of a dihedral group or its subgroup will be tripartite graph or bipartite graph, depending on n. For the next research, it is interesting to see if n is multiplication more than two different prime numbers.

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