BAREKENG: Jurnal Ilmu Matematika dan Terapan
June 2022 Volume 16 Issue 2 Page 393–408
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol16iss2pp393–408

TRACE OF THE ADJACENCY MATRIX $n \times n$ OF THE CYCLE GRAPH TO THE POWER OF TWO TO FIVE

Fitri Aryani[1*], Dian Ayu Puspita[2], Corry Corazon Marzuki[3], Yuslenita Muda[4]

[1,2,3,4] Department of Mathematics, Faculty of Science and Technology,
State Islamic University of Sultan Syarif Kasim Riau
HR. Soebrantas St., No.155 Simpang Baru, Panam, Pekanbaru, 28293

Corresponding author's e-mail: [1]khodijah_fitri@uin-suska.ac.id*

Abstract. *The main aim of this research is to find the formula of the trace of adjacency matrix $n \times n$ from a cycle graph to the power of two to five. To obtain the general form, the first step is finding the general formula of the adjacency matrix from a cycle graph to the power of two to five. Furthermore, the formula of the trace of adjacency matrix which is mentioned above obtained and proven by direct proof. We also present an implementation of the formula which is given by an example.*

Keywords: *adjacency matrix, cycle graph, direct proof, trace matrix.*

Article info:

Submitted: 16th November 2021 *Accepted: 25th March 2022*

How to cite this article:
F. Aryani, D. A. Puspita, C. C. Marzuki and Y. Muda, "TRACE OF THE ADJACENCY MATRIX $n \times n$ OF THE CYCLE GRAPH TO THE POWER OF TWO TO FIVE", *BAREKENG: J. Il. Mat. & Ter.,* vol. 16, iss. 2, pp. 393-408, June, 2022.

This work is licensed under a Creative Commons Attribution-ShareAlike 4.0 International License.
Copyright © 2022 Fitri Aryani, Dian Ayu Puspita, Corry Corazon Marzuki, Yuslenita Muda.

1. INTRODUCTION

Calculating the trace of matrix to the power is performed by multiplying the matrix or multiplying the matrix n times. After that, the trace matrix to the power will be obtained. This means that to calculate the trace of matrix to the power is quite complicated if the matrix raised to a large power. This is quite interesting to study, namely how to find the right general form to calculate the trace matrix to the power without calculating exponents or matrix multiplication. By simply substituting the matrix entries into the general form, the trace value of the power matrix will be obtained, without a long process of multiplying or matrix multiplication.

Determining the trace matrix to the power has been done by many previous researchers. In 1976 [1] has obtained an algorithm for calculating a trace matrix to the power of $Tr(A^k)$, where k is an integer and A is a Hassenberg matrix with codiagonal units. Furthermore, in 1985 [2] discussed the symbolic calculation of the trace of tridiagonal matrix with powers. The discussion of traces is also found in several applications in matrix theory and numerical linear algebra. [3], in 1990, explained in his paper that he was able to determine the eigenvalues of a symmetric matrix, also provided a basic procedure for estimating traces $(A^n)$and (A^{-n}) where n is an integer.

According to [4] in 2008 on number theory and combinatorics, trace matrices to the power of integers are associated with Euler congruences as follow:

$$Tr(Ap^r) = Tr(Ap^{r-1}) \bmod (p^r)$$

for all integer matrices A, p is a prime number and r is an integer. The article also discusses the invariant in dynamic systems which is described as a trace matrix form to the power of integers. The example given in the article is the Lefschetz number. According to [5] in 2010, the most essential problem in network analysis, specifically triangle counting in a graph, is calculating the total number of triangles in a basic connected graph while analyzing a complex network. The number is equal to $Tr(A^3)/6$, where A is the adjacency matrix of the graph. According to [6] in 2012, traces of exponent matrices are often discussed in several mathematical fields, such as Network Analysis, Number Theory, System Dynamics, Matrix Theory and Differential Equations.

The calculation of the power matrix trace has been discussed by [7] in 2015 with a matrix of order 2×2 with positive integer powers. Two general types of trace matrices of the power of positive integers were derived in this article. First, the general form of a trace matrix to the power of positive integers for even n is:

$$tr(A^n) = \sum_{r=0}^{n/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]\cdots\big[n-\big(r+(r-1)\big)\big]\big(det(A)\big)^r\big(tr(A)\big)^{n-2r}$$

Second, the general form of a trace matrix to the power of positive integers for odd n is:

$$tr(A^n) = \sum_{r=0}^{n-1/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]\cdots\big[n-\big(r+(r-1)\big)\big]\big(det(A)\big)^r\big(tr(A)\big)^{n-2r}$$

In 2017, [9] discussed the trace matrix of the order of 2×2 to the power of negative integers. In this article, there are two general forms of exponent trace matrix, provided that the determinant of the matrix is not zero. First, the general form of a trace matrix to the power of negative integers for even n is:

$$tr(A^{-n}) = \frac{\sum_{r=0}^{n/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]\cdots\big[n-\big(r+(r-1)\big)\big]\big(det(A)\big)^r\big(tr(A)\big)^{n-2r}}{\big(det(A)\big)^n}$$

Second, the general form of a trace matrix to the power of negative integers for odd n is:

$$tr(A^{-n}) = \frac{\sum_{r=0}^{n-1/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]\cdots\big[n-\big(r+(r-1)\big)\big]\big(det(A)\big)^r\big(tr(A)\big)^{n-2r}}{\big(det(A)\big)^n}$$

Furthermore, [8] did study on the trace matrix with power once again. The matrix in this article, on the other hand, is the adjacency matrix of a complete graph. The general form of the trace of adjacency matrix $n \times n$ from a complete graph to the power of an even positive integer and an odd positive integer obtained in this study is as follows:

$$tr(A_n^k) = \sum_{r=1}^{n/2} S(k,r) n(n-1)^r (n-2)^{k-2r} \qquad , k \text{ positive even number}$$

$tr\left(A_n^k\right) = \sum_{r=1}^{(n-1)/2} S(k,r)n(n-1)^r(n-2)^{k-2r}$, k odd positive number

$S(k,r)$ is a number that depends on k and r defined by:

$S(k,r) = 1, S\left(k,\frac{k}{2}\right) = 1, S\left(k,k-\frac{1}{2}\right) = \frac{k-1}{2}, S(k,r) = S(k-1,r) + S(k-2,r-1).$

In 2019, Pahade and Jha's research was devoloped by [10] with examined at the powers of negative two, three, and four in the same matrix. The typical form of the $n \times n$ adjacency matrix trace from a complete graph to the power of negative two, negative three, and negative four obtained from this study is:

$$tr(A_n^{-2}) = \frac{n\left((n-1)+(n-2)^2\right)}{(n-1)^2} \quad , n \geq 2.$$

The general form of the n ×n adjacency matrix trace of a complete graph to the negative power of three of a complete graph is:

$$tr(A_n^{-3}) = \frac{n-2(n-1)(n-2)-(n-2)^3}{(n-1)^3} \quad , n \geq 2.$$

and the general form of the n×n adjacency matrix trace to the negative power of four from a complete graph is:

$$tr(A_n^{-4}) = \frac{n((n-1)^2+3(n-1)(n-2)^2+(n-2)^4)}{(n-1)^4} \quad , n \geq 2.$$

In addition to the complete graph, there is a cycle graph which can also be represented in the adjacency matrix. A cycle graph is a graph in which every vertex has degree two and a path that starts and ends at the same vertex [8]. Thus, each cycle graph of C_n has n vertices and can be represented as a adjacent matrix of C of size $n \times n$. The general form of the adjacency matrix $n \times n$ of the cycle graph C_n is shown in Equation (1) as follows:

$$C_n = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \quad (1)$$

Based on the description of the results of previous studies regarding the trace of matrix to the power, this article will discuss trace of the adjacency matrix $n \times n$ from a cycle graph to the power of positive integers two to five.

2. RESEARCH METHODS

The method used in this research was a literature review. The following stages are mentioned in this research before providing any needed literature review: If given the adjacency matrix of the cycle graph $\boldsymbol{C_n}$ in Equation (1), then to get the trace matrix which has the power of positive integers from two to five, it will first be determined the power of the matrix from two to five. After that, the trace matrix form is obtained. In more detail, the research steps are given as follows:

1. Given the adjacency matrix of the cycle graph of C_n.
2. Prove the power of A_n^2 by $C_nC_n = A_n^2$.
3. Prove the power of A_n^3 by $A_n^2C_n = A_n^3$.
4. Prove the power of A_n^4 by $A_n^3C_n = A_n^4$.
5. Prove the power of A_n^5 by $A_n^4C_n = A_n^5$.
6. Prove $tr(A_n^2)$, $tr(A_n^3)$, $tr(A_n^4)$ and $tr(A_n^5)$ by using direct proof.

7. Apply the general forms of $tr(A_n^2)$, $tr(A_n^3)$, $tr(A_n^4)$ and $tr(A_n^5)$ with some related examples.

Proof of the general form of the trace of the adjacency matrix $n \times n$ is carried out using direct proof, namely the definition of the trace matrix in [11]. The elaboration of definitions and theorems related to the rule of matrices power and trace matrices is provided in [12], [13], [14], [15], [16].

3. RESULTS AND DISCUSSION

The results of the study were obtained after following the steps described in the research method above. There are two general forms obtained, first, the general form for the adjacency matrix $n \times n$ of the cycle graph in Equation (1) to the power of positive integers from two to five. Second, the general form of the trace of the adjacency matrix $n \times n$ from cycle graph in Equation (1) has the power of positive integers from two to five.

3.1. General Form of the Adjacency Matrix $n \times n$ of a Cycle Graph to the Power of Two

Theorem 1 Given the adjacency matrix of the cycle graph in Equation (1), so it is obtained:

$$A_n^2 = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 2 \end{bmatrix}, n \geq 6.$$

or it can be written as follows:

$$A_n^2 = [a_{ij}] = \begin{cases} 2, & for\ i = j \\ 1, & for\ i = j + 2\ \ with\ j = 1,2,3,\dots,(n-2) \\ & \quad or\ i = j - 2\ \ with\ j = 3,4,5,\dots,n \\ & \quad or\ \ i = j + n - 2\ with\ j = 1,2 \\ & \quad or\ i = j - n + 2\ with\ j = n-1, n \\ 0, & \quad for\ others \end{cases}$$

Proof:
Theorem 1 will be proved by direct proof.

$$A_n^2 = C_n \cdot C_n = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The result of multiplying the matrix entries from $C_n \cdot C_n$ can be analyzed as follows:

1. Entries with a value of 2, $for\ i = j$.

If we notice, the value of the entries along the *i*-th row in the first matrix is the same as the value of the *j*-th column entries in the second matrix where there are two numbers 1 and (n-2) numbers 0 . Thus, the result of multiplying the *i*-th row in the first matrix with the j-th column in the second matrix is:

$$a_{ij} = 0\cdot 0 + 1\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-3)\ faktor} + 1\cdot 1 = 2 \text{ with } i = j = 1,2,\cdots,n.$$

2. Entries with a value of 1.
   a. Entries in $i = j + 2\ \ with\ j = 1,2,3, \dots, (n-2)$

$$a_{3,1} = 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 1\cdot 0 + \underbrace{0 + 0 + \cdots + 0}_{(n-5)\ factor} + 0\cdot 1 = 1$$

$$a_{4,2} = 0\cdot 1 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 1\cdot 0 + \underbrace{0 + 0 + \cdots + 0}_{(n-5)\ factor} = 1$$

$$a_{5,3} = 0\cdot 0 + 0\cdot 1 + 0\cdot 0 + 1\cdot 1 + 0\cdot 1 + 1\cdot 0 + \underbrace{0 + 0 + \cdots + 0}_{(n-6)\ factor} = 1$$

$$\vdots$$

$$a_{n,n-2} = 1\cdot 0 + \underbrace{0 + 0 + \cdots + 0}_{(n-5)\ factor} + 0\cdot 1 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 = 1$$

   b. Entries in $i = j - 2\ \ with\ j = 3,4,5, \dots, n$

$$a_{1,3} = 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 0\cdot 1 + \underbrace{0 + 0 + \cdots + 0}_{(n-5)\ factor} + 1\cdot 0 = 1$$

$$a_{2,4} = 1\cdot 0 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 0\cdot 1 + \underbrace{0 + 0 + \cdots + 0}_{(n-5)\ factor} = 1$$

$$a_{3,5} = 0\cdot 0 + 1\cdot 0 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 0\cdot 1 + \underbrace{0 + 0 + \cdots + 0}_{(n-6)\ factor} = 1$$

$$\vdots$$

$$a_{n-2,n} = 1\cdot 0 + \underbrace{0 + 0 + \cdots + 0}_{(n-5)\ factor} + 1\cdot 0 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 = 1$$

   c. Entries in $i = j + n - 2\ with\ j = 1,2$

$$a_{1,n-1} = 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\ factor} + 0\cdot 1 + 0\cdot 0 + 1\cdot 1 = 1$$

$$a_{2,n} = 1\cdot 1 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\ factor} + 0\cdot 1 + 0\cdot 0 = 1$$

   d. Entries in $i = j - n + 2\ with\ j = n-1, n$

$$a_{n-1,1} = 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\ factor} + 1\cdot 0 + 0\cdot 0 + 1\cdot 1 = 1$$

$$a_{n,2} = 1\cdot 1 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\ factor} + 1\cdot 0 + 0\cdot 0 = 1$$

3. For other entries are 0 namely:
   One of entries in $i = j + 3$ with $i = 1,2,\cdots,(n-3)$

$$a_{4,1} = 0\cdot 0 + 0\cdot 1 + 1\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-6)\ factor} + 0\cdot 1 = 0$$

$$a_{5,2} = 0\cdot 1 + 0\cdot 0 + 0\cdot 1 + 1\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-6)\ factor} = 0$$

$$a_{6,3} = 0\cdot 0 + 0\cdot 1 + 0\cdot 0 + 0\cdot 1 + 1\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\ factor} = 0$$

$$\vdots$$

$$a_{n,n-3} = 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-6)\ factor} + 0\cdot 1 + 0\cdot 0 + 0\cdot 1 + 1\cdot 0 + 0\cdot 0 = 0$$

And so on, for entries that have a value other than 1 and 2, the value is zero.

Thus it can be concluded from the results that the values in the $C_n \cdot C_n$ are 0, 1 and 2 which can be presented in matrix form as follows:

$$A_n^2 = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 2 \end{bmatrix}, n \geq 6, \text{ or could be written in form:}$$

$$A_n^2 = [a_{ij}] = \begin{cases} 2, & for\ \ i = j \\ 1, & for\ i = j + 2\ \ and\ j = 1,2,3, \dots, (n-2) \\ & or\ i = j - 2\ \ with\ j = 3,4,5, \dots, n \\ & or\ \ i = j + n - 2\ with\ j = 1,2 \\ & or\ i = j - n + 2\ with\ j = n - 1, n \\ 0, & for\ others \end{cases}$$

Based on the proof above, then Theorem 1 is proven.

3.2 General Form of the Adjacency Matrix $n \times n$ of a Cycle Graph to the Power of Three

Theorem 2 Given the adjacency matrix of the cycle graph in Equation (1), so it is obtained:

$$A_n^3 = \begin{bmatrix} 0 & 3 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & 3 \\ 3 & 0 & 3 & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 3 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 3 & 0 & 3 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3 & 0 & 3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 3 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 3 & 0 & 3 \\ 3 & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 3 & 0 \end{bmatrix}, n \geq 8\text{ , or could be written as follows:}$$

$$A_n^3 = [a_{ij}] = \begin{cases} 3, & for\ i = j + 1\ \ with\ j = 1,2,3, \dots, (n-1) \\ & or\ \ i = j - 1\ \ with\ j = 2,3,4, \dots, n \\ & or\ \ i = j + n - 1\ \ with\ j = 1 \\ & or\ i = j - n + 1\ \ with\ j = n \\ 1, & for\ i = j + 3\ \ with\ j = 1,2,3, \dots, (n-3) \\ & or\ i = j - 3\ \ with\ j = 4,5,6, \dots, n \\ & or\ \ i = j + n - 3\ with\ j = 1,2\ ,3 \\ & or\ i = j - n + 3\ with\ j = (n-2), (n-1), n \\ 0, & for\ others \end{cases}$$

Proof:

Theorem 2 will be proved by direct proof.

The result of multiplying the matrix entries of $A_n^2 \cdot C_n$ can be analyzed as follows:

1. For entries with a value of 3.
   a. Entries in $i = j + 1\ \ with\ j = 1,2,3, \dots, (n-1)$

$$a_{1,2} = 2 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 + \underbrace{0 \cdot 0 + \cdots + 0 \cdot 0}_{(n-5)\ factor} + 1 \cdot 0 + 0 \cdot 0 = 3$$

$$a_{2,3} = 0 \cdot 0 + 2 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 + \underbrace{0 \cdot 0 + \cdots + 0 \cdot 0}_{(n-5)\ factor} + 1 \cdot 0 = 3$$

$$a_{3,4} = 1 \cdot 0 + 0 \cdot 0 + 2 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 + \underbrace{0 \cdot 0 + \cdots + 0 \cdot 0}_{(n-5)\ factor} = 3$$

$$\vdots$$

$$a_{n-1,n} = 1 \cdot 1 + \underbrace{0 \cdot 0 + \cdots + 0 \cdot 0}_{(n-5)\ factor} + 1 \cdot 0 + 0 \cdot 1 + 2 \cdot 1 + 0 \cdot 0 = 3$$

   b. Entries in $i = j - 1\ \ with\ j = 2,3,4, \dots, n$

$$a_{2,1} = 1\cdot 2 + 0\cdot 0 + 1\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\,factor} + 0\cdot 1 + 0\cdot 0 = 3$$
$$a_{3,2} = 0\cdot 0 + 1\cdot 2 + 0\cdot 0 + 1\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\,factor} + 0\cdot 1 = 3$$
$$a_{4,3} = 0\cdot 1 + 0\cdot 0 + 1\cdot 2 + 0\cdot 0 + 1\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\,factor} = 3$$
$$\vdots$$
$$a_{n,n-1} = 1\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\,factor} + 0\cdot 1 + 1\cdot 0 + 1\cdot 2 + 0\cdot 0 = 3$$

c. Entries in $i = j - n + 1\ with\ j = n$

$$a_{1,n} = 2\cdot 1 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\,factor} + 1\cdot 1 + 0\cdot 0 = 3$$

d. Entries in $i = j + n - 1\ with\ j = 1$

$$a_{n,1} = 1\cdot 2 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\,factor} + 1\cdot 1 + 0\cdot 0 = 3$$

2. For entries valued 1

a. Entries in $i = j - 3\ with\ j = 4,5,6,\dots,n$

$$a_{1,4} = 2\cdot 0 + 0\cdot 0 + 1\cdot 1 + 1\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 = 1$$
$$\vdots$$
$$a_{n-3,n} = 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 + 2\cdot 0 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 = 1$$

b. Entries in $i = j + 3\ with\ j = 1,2,3,\dots,(n-3)$

$$a_{4,1} = 0\cdot 2 + 0\cdot 0 + 1\cdot 1 + 0\cdot 1 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 0\cdot 1 + 0\cdot 0 = 1$$
$$\vdots$$
$$a_{n,n-3} = 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 0\cdot 1 + 0\cdot 0 + 0\cdot 2 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 = 1$$

c. Entries in $i = j - n + 3\ with\ j = (n-2),(n-1),n$

$$a_{1,n-2} = 2\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 0\cdot 1 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 = 1$$
$$a_{2,n-1} = 0\cdot 0 + 2\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 0\cdot 1 + 0\cdot 0 + 1\cdot 1 = 1$$
$$a_{3,n} = 1\cdot 1 + 0\cdot 0 + 2\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 0\cdot 1 + 0\cdot 1 = 1$$

d. Entries in $i = j + n - 3\ with\ j = 1,2\ ,3$

$$a_{n-2,1} = 0\cdot 2 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 = 1$$
$$a_{n-1,2} = 0\cdot 0 + 0\cdot 2 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 + 1\cdot 1 = 1$$
$$a_{n,3} = 1\cdot 1 + 0\cdot 0 + 0\cdot 2 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 = 1$$

3. For the other entries, it is 0. The value of 0 is the many values in the matrix, so that in this proof only a few entries are taken, namely:

a. Entries in $i = j$

$$a_{1,1} = 2\cdot 0 + 0\cdot 1 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\,factor} + 1\cdot 0 + 0\cdot 1 = 0$$
$$a_{2,2} = 0\cdot 1 + 2\cdot 0 + 0\cdot 1 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\,factor} + 1\cdot 0 = 0$$
$$a_{3,3} = 1\cdot 0 + 0\cdot 1 + 2\cdot 0 + 0\cdot 1 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\,factor} = 0$$
$$\vdots$$
$$a_{n,n} = 0\cdot 1 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\,factor} + 1\cdot 0 + 0\cdot 1 + 2\cdot 0 = 0.$$

b. Entries in $i = j + 4$ with$j = 1,2,\cdots,(n-4)$

$$a_{5,1} = 0\cdot 2 + 0\cdot 0 + 0\cdot 1 + 1\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-8)\,factor} + 0\cdot 1 + 0\cdot 0 = 0$$

$$a_{6,2} = 0 \cdot 0 + 0 \cdot 2 + 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 + 1 \cdot 0 + \underbrace{0 \cdot 0 + \cdots + 0 \cdot 0}_{(n-8)\, factor} + 0 \cdot 1 = 0$$

$$\vdots$$

$$a_{n,n-4} = 1 \cdot 0 + \underbrace{0 \cdot 0 + \cdots + 0 \cdot 0}_{(n-8)\, factor} + 0 \cdot 1 + 0 \cdot 0 + 0 \cdot 2 + 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 = 0$$

Therefore, it can be concluded from the results obtained that the values in the matrix multiplication entries of $A_n^2 \cdot C_n$ are 0, 1 and 3, which can be presented in the matrix as follows:

$$A_n^3 = \begin{bmatrix} 0 & 3 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & 3 \\ 3 & 0 & 3 & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 3 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 3 & 0 & 3 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3 & 0 & 3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 3 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 3 & 0 & 3 \\ 3 & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 3 & 0 \end{bmatrix}, n \geq 8 \text{ or could be written as follows:}$$

$$A_n^3 = [a_{ij}] = \begin{cases} 3, & for\ i = j + 1\ with\ j = 1,2,3, \dots, (n-1) \\ & or\ i = j - 1\ with\ j = 2,3,4, \dots, n \\ & or\ i = j + n - 1\ with\ j = 1 \\ & or\ i = j - n + 1\ with\ j = n \\ 1, & for\ i = j + 3\ with\ j = 1,2,3, \dots, (n-3) \\ & or\ i = j - 3\ with\ j = 4,5,6, \dots, n \\ & or\ i = j + n - 3\ with\ j = 1,2,3 \\ & or\ i = j - n + 3\ with\ j = (n-2), (n-1), n \\ 0, & for\ others \end{cases}$$

Based on the proof above, Theorem 2 is proven. ■

.

3.3. General Form of the Adjacency Matrix $n \times n$ of a Cycle Graph to the Power of Four

Theorem 3 Given the adjacency matrix of the cycle graph in Equation (1), so it is obtained**:**

$$A_n^4 = \begin{bmatrix} 6 & 0 & 4 & 0 & 1 & \cdots & 0 & 1 & 0 & 4 & 0 \\ 0 & 6 & 0 & 4 & 0 & \cdots & 0 & 0 & 1 & 0 & 4 \\ 4 & 0 & 6 & 0 & 4 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 6 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 4 & 0 & 6 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & 0 & 4 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 6 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 4 & 0 & 6 & 0 & 4 \\ 4 & 0 & 1 & 0 & 0 & \cdots & 0 & 4 & 0 & 6 & 0 \\ 0 & 4 & 0 & 1 & 0 & \cdots & 1 & 0 & 4 & 0 & 6 \end{bmatrix}, n \geq 10, \text{ or could be written in form:}$$

$$A_n^4 = [a_{ij}] = \begin{cases} 6, & for\ i = j \\ 4, & for\ i = j + 2\ with\ j = 1,2,3, \dots, (n-2) \\ & or\ i = j - 2\ with\ j = 3,4,5, \dots, n \\ & or\ i = j + n - 2\ with\ j = 1,2 \\ & or\ i = j - n + 2\ with\ j = n - 1, n \\ 1, & for\ i = j + 4\ with\ j = 1,2,3, \dots, (n-4) \\ & or\ i = j - 4\ with\ j = 5,6,7, \dots, n \\ & or\ i = j + n - 4\ with\ j = 1,2,3,4 \\ & or\ i = j - n + 4\ with\ j = n-3, n-2, n-1, n \\ 0, & for\ the\ others \end{cases}$$

Proof:

Theorem 4 will be proved by direct proof.

The result of multiplying the matrix entries of $A_n^3 \cdot C_n$ can be analyzed as follows:

1. For entries that valued 6.
   Entries in $a_{i,j}$ with $i = j = 1,2,\cdots,n$
   $$a_{1,1} = 0\cdot 0 + 3\cdot 1 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 + 3\cdot 1 = 6$$
   $$\vdots$$
   $$a_{n,n} = 3\cdot 1 + 0\cdot 1 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 + 3\cdot 1 + 0\cdot 0 = 6$$
2. For entries that valued 4.
   a. Entries in $i = j - 2\ \ with\, j = 3,4,5,\dots,n$
   $$a_{1,3} = 0\cdot 0 + 3\cdot 1 + 0\cdot 0 + 1\cdot 1 + \underbrace{0 + 0 + \cdots + 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 + 3\cdot 0 = 4$$
   $$\vdots$$
   $$a_{n-2,n} = 1\cdot 1 + \underbrace{0 + 0 + \cdots + 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 + 3\cdot 0 + 0\cdot 0 + 3\cdot 1 + 0\cdot 0 = 4$$
   b. Entries in $i = j + 2\ \ with\, j = 1,2,3,\dots,(n-2)$
   $$a_{3,1} = 0\cdot 0 + 1\cdot 3 + 0\cdot 0 + 1\cdot 1 + \underbrace{0 + 0 + \cdots + 0}_{(n-7)\,factor} + 0\cdot 1 + 0\cdot 0 + 0\cdot 3 = 4$$
   $$\vdots$$
   $$a_{n-2,n} = 1\cdot 1 + \underbrace{0 + 0 + \cdots + 0}_{(n-7)\,factor} + 0\cdot 1 + 0\cdot 0 + 0\cdot 3 + 0\cdot 0 + 1\cdot 3 + 0\cdot 0 = 4$$
   c. Entries in $i = j + n - 2\, with\, j = 1,2$
   $$a_{n-1,1} = 0\cdot 0 + 1\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 + 3\cdot 1 + 0\cdot 0 + 3\cdot 1 = 4$$
   $$a_{2,n} = 1\cdot 3 + 0\cdot 0 + 0\cdot 3 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 1 + 0\cdot 0 = 4$$
   d. Entries in $i = j - n + 2\, with\, j = n-1, n$
   $$a_{1,n-1} = 0\cdot 0 + 3\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 1 + 0\cdot 0 + 3\cdot 1 = 4$$
   $$a_{2,n} = 3\cdot 1 + 0\cdot 0 + 3\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 1 + 0\cdot 0 = 4$$
3. For entries that valued 1.
   a. Entries in $i = j - 4\ \ with\, j = 5,6,7,\dots,n$
   $$a_{1,5} = 0\cdot 0 + 3\cdot 0 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 1\cdot 0 + 0\cdot 0 + 3\cdot 0 = 1$$
   $$\vdots$$
   $$a_{n-4,n} = 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 3\cdot 0 + 0\cdot 0 + 3\cdot 0 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 = 1$$
   b. Entries in $i = j + 4\ \ with\, j = 1,2,3,\dots,(n-4)$
   $$a_{5,1} = 0\cdot 0 + 0\cdot 3 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 0\cdot 1 + 0\cdot 0 + 0\cdot 3 = 1$$
   $$\vdots$$
   $$a_{n,n-4} = 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 3\cdot 0 + 0\cdot 0 + 0\cdot 3 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 = 1$$
   c. Entries in $i = j - n + 4\, with\, j = n-3, n-2, n-1, n$
   $$a_{1,n-3} = 0\cdot 0 + 3\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 0\cdot 1 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 3\cdot 0 = 1$$
   $$\vdots$$
   $$a_{4,n} = 1\cdot 1 + 0\cdot 0 + 3\cdot 0 + 0\cdot 0 + 3\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 0\cdot 1 + 1\cdot 0 = 1$$
   d. Entries in $i = j + n - 4\, with\, j = 1,2,3,4$
   $$a_{n-3,1} = 0\cdot 0 + 0\cdot 3 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 1\cdot 0 + 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 0\cdot 3 = 1$$
   $$\vdots$$
   $$a_{n,4} = 1\cdot 1 + 0\cdot 0 + 0\cdot 3 + 0\cdot 0 + 0\cdot 3 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 1 = 1$$
4. For other entries are 0 namely:
   a. Entries in $i = j - 1$ with $j = 2,3,\cdots,(n-1)$
   $$a_{1,2} = 0\cdot 1 + 3\cdot 0 + 0\cdot 1 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 + 3\cdot 0 = 0$$
   $$\vdots$$
   $$a_{n-1,n} = 0\cdot 1 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 1 + 3\cdot 0 + 0\cdot 0 + 3\cdot 0 = 0$$

Thus, it can be concluded that the values in the matrix multiplication entries of $A_n^4 \cdot C_n$ are 0, 1, 4 and 6, can be presented in the matrix as follows:

$$A_n^4 = \begin{bmatrix} 6 & 0 & 4 & 0 & 1 & \cdots & 0 & 1 & 0 & 4 & 0 \\ 0 & 6 & 0 & 4 & 0 & \cdots & 0 & 0 & 1 & 0 & 4 \\ 4 & 0 & 6 & 0 & 4 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 6 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 4 & 0 & 6 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & 0 & 4 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 6 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 4 & 0 & 6 & 0 & 4 \\ 4 & 0 & 1 & 0 & 0 & \cdots & 0 & 4 & 0 & 6 & 0 \\ 0 & 4 & 0 & 1 & 0 & \cdots & 1 & 0 & 4 & 0 & 6 \end{bmatrix}, n \geq 10, \text{ or in form:}$$

$$A_n^4 = [a_{ij}] = \begin{cases} 6, & for\ i = j \\ 4, & for\ i = j + 2\ \ with\ j = 1,2,3, \ldots, (n-2) \\ & or\ i = j - 2\ \ with\ j = 3,4,5, \ldots, n \\ & or\ \ i = j + n - 2\ with\ j = 1,2 \\ & or\ i = j - n + 2\ with\ j = n-1, n \\ 1, & for\ i = j + 4\ \ with\ j = 1,2,3, \ldots, (n-4) \\ & or\ i = j - 4\ \ with\ j = 5,6,7, \ldots, n \\ & or\ \ i = j + n - 4\ with\ j = 1,2,3,4 \\ & or\ i = j - n + 4\ with\ j = n-3, n-2, n-1, n \\ 0, & for\ the\ others \end{cases}$$

Based on the proof above, then Theorem 4 is proven. ■

3.4 General Form of the Adjacency Matrix $n \times n$ of a Cycle Graph to the Power of Five

Theorem 4 Given the adjacency matrix of the cycle graph in Equation (1) then:

$$A_n^5 = \begin{bmatrix} 0 & 10 & 0 & 5 & 0 & \cdots & 1 & 0 & 5 & 0 & 10 \\ 10 & 0 & 10 & 0 & 5 & \cdots & 0 & 1 & 0 & 5 & 0 \\ 0 & 10 & 0 & 10 & 0 & \cdots & 0 & 0 & 1 & 0 & 5 \\ 5 & 0 & 10 & 0 & 10 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 10 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 10 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 10 & 0 & 10 & 0 & 5 \\ 5 & 0 & 1 & 0 & 0 & \cdots & 0 & 10 & 0 & 10 & 0 \\ 0 & 5 & 0 & 1 & 0 & \cdots & 5 & 0 & 10 & 0 & 10 \\ 10 & 0 & 5 & 0 & 1 & \cdots & 0 & 5 & 0 & 10 & 0 \end{bmatrix}, n \geq 12, \text{ or in another form as follows}$$

$$A_n^5 = [a_{ij}] = \begin{cases} 10, & for\ i = j + 1\ \ with\ j = 1,2,3, \ldots, (n-1) \\ & or\ i = j - 1\ \ with\ j = 2,3,4,5, \ldots, n \\ & or\ \ i = j + n - 1\ with\ j = 1 \\ & or\ i = j - n + 1\ with\ j = n \\ 5, & for\ i = j + 3\ \ with\ j = 1,2,3, \ldots, (n-3) \\ & or\ i = j - 3\ \ with\ j = 4,5,6,7, \ldots, n \\ & or\ \ i = j + n - 3\ with\ j = 1,2,3 \\ & or\ i = j - n + 3\ with\ j = n-2, n-1, n \\ 1, & for\ i = j + 5\ \ with\ j = 1,2,3, \ldots, (n-5) \\ & or\ i = j - 5\ \ with\ j = 6,7,8, \ldots, n \\ & or\ \ i = j + n - 5\ with\ j = 1,2,3,4,5 \\ & or\ i = j - n + 5\ with\ j = n-4, n-3, n-2, n-1, n \\ 0, & for\ the\ other \end{cases}$$

Proof:

Theorem 4 will be proved by direct proof.

The result of multiplying the matrix entries of $A_n^4 \cdot C_n$ can be analyzed as follows:

1. For entries with a value of 10.
   a. Entries in $i = j - 1\ \ with\ j = 2,3,4,5, \ldots, n$

$$a_{1,2} = 6\cdot 1 + 0\cdot 0 + 4\cdot 1 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 1\cdot 0 + 0\cdot 0 + 4\cdot 0 + 0\cdot 0 = 10$$
$$\vdots$$
$$a_{n-1,n} = 4\cdot 1 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 4\cdot 0 + 0\cdot 1 + 6\cdot 1 + 0\cdot 0 = 10$$

b. Entries in $i = j + 1$ $with\, j = 1,2,3,\dots,(n-1)$

$$a_{2,1} = 1\cdot 6 + 0\cdot 0 + 1\cdot 4 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 0\cdot 1 + 0\cdot 0 + 0\cdot 4 + 0\cdot 0 = 10$$
$$\vdots$$
$$a_{n,n-1} = 1\cdot 4 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 0\cdot 4 + 1\cdot 0 + 1\cdot 6 + 0\cdot 0 = 10$$

c. Entries in $i = j - n + 1$ $with\, j = n$

$$a_{1,n} = 6\cdot 1 + 0\cdot 0 + 4\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 1\cdot 0 + 0\cdot 0 + 4\cdot 1 + 0\cdot 0 = 10$$

d. Entries in $i = j + n - 1$ $with\, j = 1$

$$a_{n,1} = 1\cdot 6 + 0\cdot 0 + 0\cdot 4 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 0\cdot 1 + 0\cdot 0 + 1\cdot 4 + 0\cdot 0 = 10$$

2. For entries that valued 5

a. Entries in $i = j - 3$ $with\, j = 4,5,6,7,\dots,n$

$$a_{1,4} = 6\cdot 0 + 0\cdot 0 + 4\cdot 1 + 0\cdot 0 + 1\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 1\cdot 0 + 0\cdot 0 + 4\cdot 0 + 0\cdot 0 = 5$$
$$\vdots$$
$$a_{n-3,n} = 1\cdot 1 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 6\cdot 0 + 0\cdot 1 + 4\cdot 1 + 0\cdot 0 = 5$$

b. Entries in $i = j + 3$ $with\, j = 1,2,3,\dots,(n-3)$

$$a_{4,1} = 0\cdot 6 + 0\cdot 0 + 1\cdot 4 + 0\cdot 0 + 1\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 0\cdot 1 + 0\cdot 0 + 0\cdot 4 + 0\cdot 0 = 5$$
$$\vdots$$
$$a_{n,n-3} = 1\cdot 1 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 0\cdot 6 + 1\cdot 0 + 1\cdot 4 + 0\cdot 0 = 5$$

c. Entries in $i = j - n + 3$ $with\, j = n-2, n-1, n$

$$a_{1,n-2} = 6\cdot 0 + 0\cdot 0 + 4\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 1\cdot 1 + 0\cdot 0 + 4\cdot 1 + 0\cdot 0 = 5$$
$$\vdots$$
$$a_{3,n} = 4\cdot 1 + 0\cdot 0 + 6\cdot 0 + 0\cdot 0 + 4\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 1 + 0\cdot 0 = 5$$

d. Entries in $i = j + n - 3$ $with\, j = 1,2,3$

$$a_{n-2,1} = 0\cdot 6 + 0\cdot 0 + 0\cdot 4 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 1\cdot 1 + 0\cdot 0 + 1\cdot 4 + 0\cdot 0 = 5$$
$$\vdots$$
$$a_{n,3} = 1\cdot 4 + 0\cdot 0 + 0\cdot 6 + 0\cdot 0 + 0\cdot 4 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 1 + 0\cdot 0 = 5$$

3. For entries that valued 1

a. Entries $i = j - n + 5$ $with\, j = n-4, n-3, n-2, n-1, n$

$$a_{1,n-4} = 6\cdot 0 + 0\cdot 0 + 4\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 1\cdot 1 + 0\cdot 0 + 4\cdot 0 + 0\cdot 0 = 1$$
$$\vdots$$
$$a_{5,n} = 1\cdot 1 + 0\cdot 0 + 4\cdot 0 + 0\cdot 0 + 6\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 0\cdot 1 + 0\cdot 0 = 1$$

b. Entries in $= j + n - 5$ $with\, j = 1,2,3,4,5$

$$a_{n-4,1} = 0\cdot 6 + 0\cdot 0 + 0\cdot 4 + 0\cdot 0 + 0\cdot 1 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\,factor} + 1\cdot 1 + 0\cdot 0 + 0\cdot 4 + 0\cdot 0 = 1$$
$$\vdots$$
$$a_{n,5} = 1\cdot 1 + 0\cdot 0 + 0\cdot 4 + 0\cdot 0 + 0\cdot 6 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-7)\,factor} + 1\cdot 0 + 0\cdot 0 = 1$$

4. For other entries are 0, namely:
   Entries in $a_{i,i+2}$ with $i = 1,2,\cdots,(n-2)$

$$a_{1,3} = 6\cdot 0 + 0\cdot 1 + 4\cdot 0 + 0\cdot 1 + 1\cdot 0 + \underbrace{0 + 0 + \cdots + 0}_{(n-9)\ factor} + 1\cdot 0 + 0\cdot 0 + 4\cdot 0 + 0\cdot 0 = 0$$

$$\vdots$$

$$a_{n-2,n} = 0\cdot 1 + 1\cdot 0 + \underbrace{0 + 0 + \cdots + 0}_{(n-7)\ factor} + 4\cdot 0 + 0\cdot 0 + 6\cdot 0 + 0\cdot 1 + 4\cdot 0 = 0$$

Entries in $a_{i,j}$ with $i = j = 1,2,\cdots,n$

$$a_{1,1} = 6\cdot 0 + 0\cdot 1 + 4\cdot 0 + 0\cdot 0 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-9)\ factor} + 1\cdot 0 + 0\cdot 0 + 4\cdot 0 + 0\cdot 1 = 0$$

$$\vdots$$

$$a_{n,n} = 0\cdot 1 + 1\cdot 0 + \underbrace{0\cdot 0 + \cdots + 0\cdot 0}_{(n-5)\ factor} + 1\cdot 0 + 0\cdot 1 + 2\cdot 0 = 0$$

Thus, it can be concluded that the values in the matrix multiplication entries of $A_n^4 \cdot C_n$ are 0, 1, 5 and 10, which can be written in the matrix as follows:

$$A_n^5 = \begin{bmatrix} 0 & 10 & 0 & 5 & 0 & \cdots & 1 & 0 & 5 & 0 & 10 \\ 10 & 0 & 10 & 0 & 5 & \cdots & 0 & 1 & 0 & 5 & 0 \\ 0 & 10 & 0 & 10 & 0 & \cdots & 0 & 0 & 1 & 0 & 5 \\ 5 & 0 & 10 & 0 & 10 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 10 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 10 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 10 & 0 & 10 & 0 & 5 \\ 5 & 0 & 1 & 0 & 0 & \cdots & 0 & 10 & 0 & 10 & 0 \\ 0 & 5 & 0 & 1 & 0 & \cdots & 5 & 0 & 10 & 0 & 10 \\ 10 & 0 & 5 & 0 & 1 & \cdots & 0 & 5 & 0 & 10 & 0 \end{bmatrix}$$, $n \geq 12$ or could be written in another form, namely

$$A_n^5 = [a_{ij}] = \begin{cases} 10, & for\ i = j+1\ \ with\ j = 1,2,3,\dots,(n-1) \\ & or\ i = j-1\ \ with\ j = 2,3,4,5,\dots,n \\ & or\ \ i = j+n-1\ with\ j = 1 \\ & or\ i = j-n+1\ with\ j = n \\ 5, & for\ i = j+3\ \ with\ j = 1,2,3,\dots,(n-3) \\ & or\ i = j-3\ \ with\ j = 4,5,6,7,\dots,n \\ & or\ \ i = j+n-3\ with\ j = 1,2,3 \\ & or\ i = j-n+3\ with\ j = n-2, n-1, n \\ 1, & for\ i = j+5\ \ with\ j = 1,2,3,\dots,(n-5) \\ & or\ i = j-5\ \ with\ j = 6,7,8,\dots,n \\ & or\ \ i = j+n-5\ with\ j = 1,2,3,4,5 \\ & or\ i = j-n+5\ with\ j = n-4, n-3, n-2, n-1 \\ 0, & for\ the\ other \end{cases}$$

Based on the proof above, then Theorem 4 is proven. ■

3.5 General Form of Trace of the Adjacency Matrix $n \times n$ of Cycle Graph

The next result of this research is that by using the results from Theorem 2 to Theorem 5, then the trace of the adjacency matrix $n \times n$ of the cycle graph can be obtained and presented in Corollary 1 to Corollary 4.

Corollary 1 Given the adjacency matrix $n \times n$ of the cycle graph expressed in Equation (1.6), so it is obtained:

$$tr(A_n^2) = 2n \qquad n \geq 6$$

Proof:

By using the definition of the trace matrix and the results of the research in Theorem 2, it is obtained:

$$tr(A_n^2) = tr\left(\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 2 \end{bmatrix}\right)$$

$$= \underbrace{2 + 2 + \cdots + 2}_{n}$$

$$= 2n$$

Based on the proof above, Corollary 6 is proven. ■

Corollary Given the adjacency matrix $n \times n$ of the cycle graph expressed in Equation (1.9) then it is obtained:

$$tr(A_n^3) = 0 \qquad n \geq 8$$

Proof:

By using the definition of the trace matrix and Theorem 3, it is obtained:

$$tr(A_n^3) = tr\left(\begin{bmatrix} 0 & 3 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & 3 \\ 3 & 0 & 3 & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 3 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 3 & 0 & 3 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3 & 0 & 3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 3 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 3 & 0 & 3 \\ 3 & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 3 & 0 \end{bmatrix}\right)$$

$$= \underbrace{0 + 0 + \cdots + 0}_{n}$$

$$= 0n$$

$$= 0$$

Based on the proof above, then Corollary 2 is proven. ■

Corollary 3 Given the adjacency matrix $n \times n$ of the cycle graph stated in Equation (1.6) then:

$$tr(A_n^4) = 6n \qquad n \geq 10$$

Proof:

By using the definition of the trace matrix and Theorem 3, then it is obtained: $tr(A_n^4) =$

$$tr\left(\begin{bmatrix} 6 & 0 & 4 & 0 & 1 & \cdots & 0 & 1 & 0 & 4 & 0 \\ 0 & 6 & 0 & 4 & 0 & \cdots & 0 & 0 & 1 & 0 & 4 \\ 4 & 0 & 6 & 0 & 4 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 6 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 4 & 0 & 6 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & 0 & 4 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 6 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 4 & 0 & 6 & 0 & 4 \\ 4 & 0 & 1 & 0 & 0 & \cdots & 0 & 4 & 0 & 6 & 0 \\ 0 & 4 & 0 & 1 & 0 & \cdots & 1 & 0 & 4 & 0 & 6 \end{bmatrix}\right)$$

$$= \underbrace{6 + 6 + \cdots + 6}_{n}$$

$$= 6n$$

Based on the proof above, Corollary 3 is proven. ■

Corollary 4 Let the adjacency matrix $n \times n$ of the cycle graph stated in Equation (1.6), then:

$$tr(A_n^5) = 0 \qquad n \geq 8$$

Proof:

By using the definition of the trace matrix and Theorem 4, then the following is obtained:

$$tr(A_n^5) = tr\left(\begin{bmatrix} 0 & 10 & 0 & 5 & 0 & \cdots & 1 & 0 & 5 & 0 & 10 \\ 10 & 0 & 10 & 0 & 5 & \cdots & 0 & 1 & 0 & 5 & 0 \\ 0 & 10 & 0 & 10 & 0 & \cdots & 0 & 0 & 1 & 0 & 5 \\ 5 & 0 & 10 & 0 & 10 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 10 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 10 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 10 & 0 & 10 & 0 & 5 \\ 5 & 0 & 1 & 0 & 0 & \cdots & 0 & 10 & 0 & 10 & 0 \\ 0 & 5 & 0 & 1 & 0 & \cdots & 5 & 0 & 10 & 0 & 10 \\ 10 & 0 & 5 & 0 & 1 & \cdots & 0 & 5 & 0 & 10 & 0 \end{bmatrix}\right)$$

$$= \underbrace{0 + 0 + \cdots + 0}_{n}$$

$$= 0$$

Based on the proof above, Corollary 4 is proven. ■

3.6 Application to the Power and Trace of an Adjacency Matrix $n \times n$ of a Circle Graph to the Power of a Positive Integer

The following is an example which uses the effects trace of the adjacency matrix $n \times n$ of a cycle graph to the power of positive integers.

Example : Let a circle graph of C_{15} as follows:

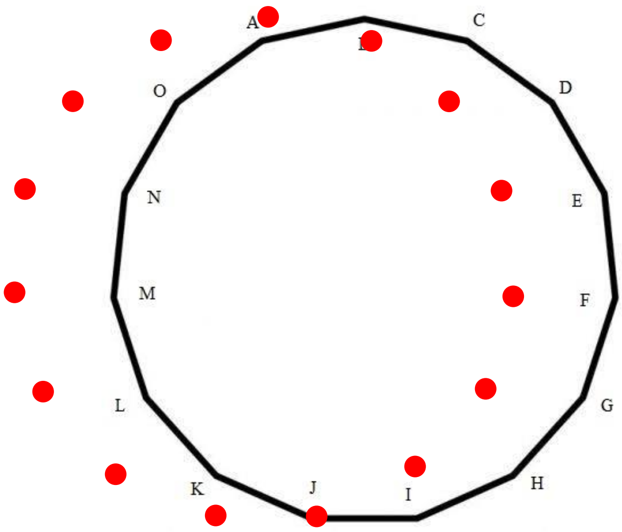

Figure 1. Graph of Cyle C_{15}

Determine the adjacency matrix of the circle graph and the power of the adjacency matrix with powers of 2,3,4 and 5 and determine $tr(A_{15}^2)$, $tr(A_{15}^3)$, $tr(A_{15}^4)$ and $tr(A_{15}^5)$ by using the existing theorems!

Answers:

a. Based on the Figure 1 above, then the adjacency matrix of the cycle graph above is as follows:

$$C_{15} = \begin{matrix} & A & B & C & D & E & F & G & H & I & J & K & L & M & N & O \\ A & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ B & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ D & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ G & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ H & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ J & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ M & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ N & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ O & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}$$

b. Based on the adjacency matrix in (a), it is known that n=15, so by using Corollary 1, 2, 3 and 4, the following is obtained:

1. $tr(A_{15}^2) = 2(15) = 30$
2. $tr(A_{15}^3) = 0$
3. $tr(A_{15}^4) = 6(15) = 90$
4. $tr(A_{15}^5) = 0$

4. CONCLUSIONS

Based on the discussion described above, regarding the trace of the adjacency matrix $n \times n$ to the power of positive integers from a cycle graph with the matrix form in Equation (1), the following conclusions are obtained:

1. The general forms of the adjacency matrix $n \times n$ of a cycle graph to the power of positive integers two to five are:

$$A_n^2 = [a_{ij}] = \begin{cases} 2, & for\ i = j \\ 1, & for\ i = j + 2\ \ with\ j = 1,2,3,\ldots,(n-2) \\ & or\ i = j - 2\ \ with\ j = 3,4,5,\ldots,n \\ & or\ \ i = j + n - 2\ with\ j = 1,2 \\ & or\ i = j - n + 2\ with\ j = n - 1, n \\ 0, & for\ the\ others \end{cases}$$, and

$$A_n^3 = [a_{ij}] = \begin{cases} 3, & for\ i = j + 1\ \ with\ j = 1,2,3,\ldots,(n-1) \\ & or\ \ i = j - 1\ \ with\ j = 2,3,4,\ldots,n \\ & or\ \ i = j + n - 1\ \ with\ j = 1 \\ & or\ i = j - n + 1\ \ with\ j = n \\ 1, & for\ i = j + 3\ \ with\ j = 1,2,3,\ldots,(n-3) \\ & or\ i = j - 3\ \ with\ j = 4,5,6,\ldots,n \\ & or\ \ i = j + n - 3\ with\ j = 1,2\ ,3 \\ & or\ i = j - n + 3\ with\ j = (n-2),(n-1),n \\ 0, & for\ the\ others \end{cases}$$, and

$$A_n^4 = [a_{ij}] = \begin{cases} 6, & for\ i = j \\ 4, & for\ i = j + 2\ \ with\ j = 1,2,3, \dots, (n-2) \\ & or\ i = j - 2\ \ with\ j = 3,4,5, \dots, n \\ & or\ \ i = j + n - 2\ with\ j = 1,2 \\ & or\ i = j - n + 2\ with\ j = n - 1, n \\ 1, & for\ i = j + 4\ \ with\ j = 1,2,3, \dots, (n-4) \\ & or\ i = j - 4\ \ with\ j = 5,6,7, \dots, n \\ & or\ \ i = j + n - 4\ with\ j = 1,2,3,4 \\ & or\ i = j - n + 4\ with\ j = n-3, n-2, n-1, n \\ 0, & for\ the\ other \end{cases}$$

, and the last

$$A_n^5 = [a_{ij}] = \begin{cases} 10, & for\ i = j + 1\ \ with\ j = 1,2,3, \dots, (n-1) \\ & or\ i = j - 1\ \ with\ j = 2,3,4,5, \dots, n \\ & or\ \ i = j + n - 1\ with\ j = 1 \\ & or\ i = j - n + 1\ with\ j = n \\ 5, & for\ i = j + 3\ \ with\ j = 1,2,3, \dots, (n-3) \\ & or\ i = j - 3\ \ with\ j = 4,5,6,7, \dots, n \\ & or\ \ i = j + n - 3\ with\ j = 1,2,3 \\ & or\ i = j - n + 3\ with\ j = n-2, n-1, n \\ 1, & for\ i = j + 5\ \ with\ j = 1,2,3, \dots, (n-5) \\ & or\ i = j - 5\ \ with\ j = 6,7,8, \dots, n \\ & or\ \ i = j + n - 5\ with\ j = 1,2,3,4,5 \\ & or\ i = j - n + 5\ with\ j = n-4, n-3, n-2, n-1, n \\ 0, & for\ the\ others \end{cases}$$

2. The general forms trace of the adjacency matrix $n \times n$ of a cycle graph to the power of positive integers two to five are:
   a. $tr(A_n^2) = 2n \qquad , n \geq 6$
   b. $tr(A_n^3) = 0n \qquad , n \geq 8$
   c. $tr(A_n^4) = 6n \qquad , n \geq 10$
   d. $tr(A_n^5) = 0n \qquad , n \geq 12$

REFERENCES

[1] Data, B.N, dan Datta, K. An Algorithm for Computing Power of a Hessenberg Matrix and its Applictions, *Linear Algebra and its Application*, 14, 273-284. 1976.

[2] Chu, M.T, and Raleigh. Symbolic Calculation of the Trace of the Power of a Tridiagonal Matrix, *Computing*, 35, 257-268. 1985.

[3] Pan, V. Estimating the Extremal Eigenvalues of a Symetric Matrix, *Computers & Mathematics with Applications*, 20, 17-22. 1990.

[4] Zarelua, A.V. "On Congruences for the Trace of Power of Some Matrices". *Proceedings of the Steklov Institute of Mathematics*, 263, 78-98, 2008.

[5] Avron, H., Counting Triangles in Large Graphs Using Randomized Matrix Trace Estimation. *Proceeding of Kdd-Ldmta'10*, 2010.

[6] Brezinski, C, P.Fika dan M.Mitrouli, Estimations of the Trace of Power of Positive by Extrapolation of the Moment, *Electronic Transactions on Numerical Analysis*, 39, 144-155, 2012.

[7] Pahade, J., and M. Jha, Trace of Positive Integer Power of Real 2×2 matrices, *Advancesin Linear Algebra & Matrix Theory*, 5, 150-155, 2015.

[8] Pahade, J., and M. Jha, Trace of Positive Integer Power of Adjacency Matrix, Global Journal of Pure and Applied Mathematics, Vol 13 (6), 2017

[9] Aryani, F, dan Solihin, M. Trace Matriks Real Berpangkat Bilangan Bula Negatif, *Jurnal Sains Matematika dan Statistika*, Vol.3 (2), 2017.

[10] F. Aryani, dkk, "Trace Matriks Ketetanggan n x n Berpangkat -2, -3, -4" Proceding SNTIKI 12, hal 543-553, 1 Desember 2020.

[11] K. H. Rosen, *Discrete Mathematics and Its Applications*. New York: Mc Graw Hill, 2007

[12] R. Munir, *Matematika Disktrit Edisi Ketiga*. Bandung: Informatika, 2005.

[13] M. dan Marjono, *AljabarLinear*. Malang: UB Press, 2012.

[14] H. A. dan C. Rorres, *Elementary Linear Algebra*. Wiley: United States of Amerika, 2013.

[15] R. Rifa'i, *Aljabar Matriks Dasar*. Yogyaka: Budi Utama, 2016.

[16] H. A. dan C. Rorres, *Aljabar Linear Elementer Versi Aplikasi Edisi Kedelapan*. Jakarta: Erlangga, 2004