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Analysis of The Occurrence of Reversible Reasoning for Inverse Cases: A Case Study on The Subject Adjie

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Abstract

Background. Student reasoning in learning mathematics contributes significantly to the achievement of student mathematics learning outcomes. **The main objective** of this study is to investigate the process of reversible reasoning in students for inverse problems, in the case of Adjie (Ad). **The research method** used to reveal the reversible reasoning in Adjie's case using descriptive qualitative research methods. **Sampling** was carried out using purposive sampling technique where the research sample was selected based on reversible reasoning criteria. **Retrieval research data** uses the results of students' mathematical work, think aloud, interviews, and the components that cause reversible reasoning. **The results** of our study found that the process begins with an obstacle that causes Ad to be unable to continue the resolution process, resulting in a metacognition process by analyzing the problem again analytically and developing other heuristic strategies. Ad shows a change in perspective where he initially interpreted inverse as the act of swapping independent and dependent variables and switched to interpreting inverse as the opposite of a function process involving analogy and image representation. **The contribution** of this research provides knowledge that reversible reasoning can occur in understanding and solving mathematical problems in inverse material.

Keywords: *Reversible Reasoning, Inverse Problem, Metacognition, Refraction, Analytic, Intuitive*

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1. Introduction

Currently, the study of reversible reasoning has made significant developments and is one of the most interesting topics in recent years. This is due to the awareness that reversible reasoning is related to many problems in mathematics, for example, in operational problems (addition, subtraction, multiplication, and division) that involve knowledge of algebra and fractions (Hackenberg, 2013; Hackenberg & Lee, 2016; Lee & Hackenberg, 2014). Several findings indicate the importance of reversible reasoning in minimizing learning disabilities in the learning practice of both students and teachers (Di Stefano, Litster, & MacDonald, 2017; B. Dougherty, Bryant, Bryant, & Shin, 2016; BJ Dougherty, Bryant, Bryant, Darrough, & Pfannenstiel, 2015), learning strategy development (Haciomeroglu, Aspinwall, & Presmeg, 2009, 2010; Mackrell, 2011; Nolte & Pamperien, 2017; Vilkomir & O'Donoghue, 2009), process-based assessment (Sangwin & Jones, 2017), and the demands of the curriculum (Linchevski & Herscovics, 1996; Lubin, Simon, Houdé, & De Neys, 2015). So that it is not a new thing in mathematics education research.

The term Reversible Reasoning was originally based on Piaget's and Krutetskii's concept of reversibility. Piaget (Inhelder & Piaget, 1958) revealed that reversibility is the mental ability of students to change the direction of their thinking back to the starting point. Operationally, he distinguishes reversibility in two forms, namely negation and reciprocity. Negation expresses the idea that every operation has an opposite, say addition and subtraction, whereas reciprocity refers to the coordination between two sides of a relationship, for example, if $a > b$ then $b < a$ or if $a = b$ then $b = a$. Meanwhile, Krutetskii (Krutetskii, 1976) reveals reversibility as a mental process in reconstructing the direction of thinking that shifts from direct thinking to the opposite. Krutetskii argues that if students learn the six stages (A, B, C, D, E, F), then that student has learned the process from A to F. However, reversing the process from F to A does not need to follow the order, but reversible reasoning only requires a process starting at F and deducing A.

Since Piaget and Krutetskii introduced the concept of reversibility. The research community has attempted to break down reversibility into several forms. Ramful (Ramful, 2009, 2014, 2015; Ramful & Olive, 2008) argues that reversible reasoning is the mental flexibility of an individual who is able to anticipate and revisit problems analytically. He adds that the problem situation given is a trigger that causes students to do reversible reasoning. In comparison, Hackenberg (Hackenberg, 2010; Hackenberg & Lee, 2015, 2016) reveals that reversible reasoning is related to anticipatory schemes where students try to achieve the results of previous experiences by producing the cause of a problem. Being able to produce the causes of an outcome tends to require analytical reflection so that students experience changes in their point of view. In this article, we define reversible reasoning as a mental process of changing a point of view based on a given problem situation.

We highlight Hackenberg's opinion about the anticipation scheme (in von Glasersfeld's schema theory as the expectation that the activity produces a previously experienced result) as a precondition for reversible reasoning (Hackenberg, 2010). The anticipation scheme is considered a cause-and-effect relationship in which students restate previous experiences, reflect on them, describe regularities, and project these as predictions that may be repeated (Ron Tzur, 2011; Simon, Kara, Placa, & Sandir, 2016; Simon, Placa, & Avitzur, 2016; Tzur, 2007; Tzur & Simon, 2004). The anticipation scheme can be divided into three: first, the anticipation scheme does not require causal rules, but depends on familiarity; second, the anticipation scheme by predicting the results based on the consequences previously experienced; and third, the anticipation scheme related to consequences from previous experiences by explaining causal relationships (Simon, Kara, et al., 2016; Simon, Placa, et al., 2016).

The third anticipation scheme causes reversible reasoning because students try to achieve the results of previous experiences by generating the cause. Attempts to achieve the results of previous experiences require mental activity through reflection and require triggers that cause them to reflect. When students' cognition asks themselves why they experience a certain effect (effect), they can abstract the cause of the effect by separating the factors that produce the effect while paying attention to changes caused by the selected cause (Chun, 2017). We conclude that the general picture of reasoning is reversible, namely: students recursively use the results of the scheme to produce causes, build (to the project) causal relationships into new experiences, separate (to isolate) causes in order to obtain results by reconstructing problem given.

Reconstructing mental processes, changing the direction of thinking (direct direction towards the opposite), and developing a two-way relationship requires certain types of questions to express (Flanders, 2014). Some researchers developed two paired problem models (direct and reverse problems), which are used interchangeably to investigate whether students are able to infer relationships from the problem (Flanders, 2014; Krutetskii, 1976; Lee & Hackenberg, 2014; Ramful, 2014; Simon, Kara, et al., 2016; Leslie P Steffe, 2002). The problem with inverted questions develops students' active thinking, although sometimes it requires a process to solve it due to the cognitive gaps experienced by students (Linchevski & Herscovics, 1996; Nathan & Koedinger, 2000; Vergnaud, 1998). Reversible reasoning can be revealed by developing problems that can relate two concepts whose processes and rules can be used interchangeably and problems that can be presented with multiple representations.

The literature review on reversible reasoning is limited to finding missing values in the problem of addition and subtraction, multiplication or division, or unknown quantities or finding other forms of equations by developing algebraic and fraction concepts (Hackenberg, 2010; Hackenberg & Lee, 2015; Ramful, 2009, 2015; Ramful & Olive, 2008; Lesllie P. Steffe & Olive, 2009; Tzur, 2004). However, we consider that there are still few studies that reveal the causes for reversible reasoning. In addition, no one has linked reversible reasoning, especially to mathematical problems at the university level, for example, building exponential and logarithmic relationships, finding θ in trigonometric functions, sketching antiderivative graphs based on known derivatives, and the relationship between functions and inverses. This is important to develop in this research because the problems at the university level are more complex than at the basic level.

In this article, we focus on inverse problems. Based on the initial hypothesis, we suspect that the cause of students solving problems with reversible reasoning for inverse problems, namely: when they experience obstacles or are unable to continue the problem-solving process, students try to find other alternatives (rebuild the prerequisite concepts and previous experiences). This is a development of Ramful's findings (Ramful, 2009, 2014, 2015) that reversible reasoning occurs based on problem situations, so it needs to be explored in depth through case studies by analyzing work results, thinking aloud, interviewing, and revealing the components that cause reversible reasoning.

We identify two facts based on the limitations of previous research. First, there is still little study of reversible reasoning at the university level; second, there is no literature that reveals the process that causes students to solve problems by doing reversible reasoning, and this can be traced through the initial symptoms obtained in previous studies. These facts lead us to fill the gap in this article with a research question: how does reversible reasoning occur? It is intended that the findings can contribute to understanding the development of reversible reasoning in students and promote characteristics of reversible reasoning that can be useful for long-term research. Although we limit our focus to student and function problems and inverse, we argue that this is an important step to provide room for the development of reversible reasoning in classroom learning.

2. Research Method

In this study, a qualitative method with a case study approach was used for data collection and analysis. The reason for choosing this method is to understand, recognize phenomena and reveal unique things from student behavior in solving problems. Our analysis was obtained from a series of interviews with one student, Adjie (Ad). He was chosen from 34 students enrolled in the same class and was 19 years old. The reasons for his selection were (1) during the initial instrument administration, he encountered obstacles and was unable to continue the process based on his initial expectations, but was trying to find a solution; (2) he was able to reconstruct previous knowledge and used it to solve a given problem; and (3) he was able to communicate the ideas that arise in his mind. Based on the preliminary analysis, we suspect that the Ad conception can provide insight into the process of reversible reasoning.

2.1 Task Development

To get an overview of the participant's reversible reasoning process, we ask Ad to rework the task we developed by modifying the instruments from previous research (Figure 1). We use the Paoletti instrument (Paoletti, 2015; Paoletti, Stevens, Hobson, Moore, & LaForest, 2018) for task 1 using the polynomial function. We distinguish these assignments from the tasks used during the initial investigation because we want participants to move out of situations. They often encounter and generate reversible reasoning in a natural way. We present a problem with a new situation that allows them to reverse the thinking of a problem situation that is different from the usual, developing their thinking on a problem with a new situation, and changing their perspective.

Task	Task Characteristic	Possible Solution
Task #1: If $f(x) = \frac{x^2+2x+6}{x^2+x-5}$ and $f^{-1}(2) = x$. Explain how do you determine the x value?	Students need to understand the relationship between $f(x) = y$ equal with $f^{-1}(y) = x$	The task asks x value of $f^{-1}(2) = x$ equal with $f(x) = 2$.

Figure 1. Task Development

2.2 Data Collection

Task-based interviews are used to collect data about the process of reversible reasoning (Goldin, 2000). We emphasize on the subject that we do not care about whether the answer is right or wrong, and we ask them to "think aloud" while completing the task (Creswell, 2012; Miles, Huberman, & Saldana, 2014). Interviews are conducted for 30-60 minutes and are conducted every weekend. Ad is asked to answer all assignments and answer additional questions posed by the researcher regarding the reasons, strategies used, and things that help them while completing the task. During the interview, they are free to express their responses regarding the assignment given. The interview is recorded using audio and video for further analysis.

2.3 Technique of Data Analysis

Prior to conducting the analysis, we provided an overview of the development of an analytical framework used to encode and analyze data sources. The coding developed used student thought data. We began data analysis after all interview sessions and used initial coding of the recorded interviews. We listened to recorded interviews, work results and transcribed them as needed. The focus of this coding is what causes them to change the point of view of a given problem situation. All interview data were collected, audio and video recordings were transcribed, and copies of student

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work were combined with each transcript. All forms of data were analyzed systematically and independently coded according to the characteristics of students' reversible reasoning (Creswell, 2012; Miles et al., 2014.). Student assignments were analyzed by determining whether Ad had successfully completed each assignment, identifying the reasons for their changing their point of view, and the strategies used in changing that point of view.

For data analysis, we used the thematic analysis method developed by Braun & Clarke (Braun & Clarke, 2006). The aim was to identify patterns of themes through a series of existing data to answer the research questions posed. Patterns of themes are identified through a meticulous process of data recognition (data familiarization), data coding, and theme development and revision. This method is organized into six stages (Braun & Clarke, 2006). First, reading the interview transcript over and over again. Second, developing the initial code by looking for words or phrases that indicate that participants change their point of view from a given problem situation. Third, defining the theme by creating, defining, modifying the code to understand the relationships and develop potential themes. Based on the initial code, we compare related quotes to find themes between the codes.

Fourth, reviewing the theme by identifying themes in the previous stage, discussing their suitability with the data, discussing themes. Some themes underwent changes in naming or descriptions. In addition, we removed themes that had no evidence to prove that the participants did reversible reasoning. Fifth, defining and naming themes, we define the theme by entering the main ideas and making a description of each theme (can be seen in Table 1). This section is very important for answering research questions. and Sixth, producing reports.

Table 1. Theme development of reversible reasoning

Theme	Description	Phrase Used
Intuitive	intuitively in concluding the problems faced based on the knowledge they have or previous experience	“this problem using this concept”
Analytic	• Exploration reduce the complexity of the problem, and the results of the reduction are used to draw conclusions	“I try to change the problem like this”
	• Convergent / Declination Identify special cases (out of context of the problem), and the results are used to compare with the problem at hand.	“suppose I take another example”
Refraction	changes in point of view that occur due to obstacles, being unable to continue the problem-solving process, and the presence of metacognitive thinking to develop heuristic strategies	“it looks like something is wrong”

Research Result

To answer the research question, we first discussed Ad's response to task 1. Based on these responses, we developed a reversible reasoning model Ad by investigating the problem solving developed by Yimer & Ellerton (Yimer & Ellerton, 2010), namely how he analyzes problems, formulates the results of the analysis for further exploration, implementation, evaluation, and reflection. Our focus is whether Ad has an anticipatory scheme in restating his previous experiences through reflection, what causes him to reflect and ultimately changes his point of view. This can answer the research question: how is the process of reversible reasoning occurring.

Ad's solution for inverse problem

Table 2 shows a quote from Ad about the inverse problem related to the behavior and problem-solving phase, and Figure 2 shows the results of the solution. Ad started by reading problem (1) and restated problem (2), thereby building an understanding of the problem. He revealed that he had encountered this kind of problem before (3) and underlined the main idea of the problem (4). The results of previous experience were used to investigate the problem (5), namely finding the inverse by replacing $f(x)$ with y. After analyzing the problem, he implemented it through procedural operations by multiplying and grouping similar variables with the x^2 and x variables on the left of the equal sign, while the constant variables were on the right (6 - 7). He tried to remove the x variable, but he encountered a problem (8). This caused him to reexamine the completion steps taken (9). Then he conjectured that by moving the constant variable to the left of the equal sign, the problem was in the form of a quadratic equation and could be found by factoring (10 - 12). He tried to factor it, but this was unsuccessful because the form of the quadratic equation obtained was too complex, so he experienced obstacles to continuing the solving process (13). He reflected back on the original goal by using a strategy of replacing (x) with y based on previous experience (14).

He felt challenged to solve the problem by rereading the problem (15), recalling the problems that have been solved previously (16), and concluding the level of difficulty of the problem faced compared to the previous problem (17). He underlined and highlighted the inverse as the main idea and tried to find the relationship between the function and the inverse by identifying special cases outside the problem, for example, for $f(x) = x + 1$ and $f(x) = x^2$ (18 – 20). The phrase used in identifying the case is "if $f^{-1}(2)$ from this f [refers to $f(x) = x + 1$], then x is equal to 1", "if $f(1)$ means the result is 2", As well as in other cases (for $f(x) = x^2$), it is done repeatedly. The results of the identification of these cases found an initial assumption that "how many x produces 2" (21). He tried to prove the conjecture compared to the problem at hand, justifies it by visualizing it in a venn diagram (22 - 23), and finally found the relationship between the function and the inverse used to solve the problem through procedural operations and found a solution (25 - 28).

When he tried to justify the solution, he was not sure about the solution he got, so he reflected by substituting the value of x, and both of them produce 2 (29 - 30). He expressed excitement about how difficult the problem was and did not expect the problem-solving process to be carried out as such

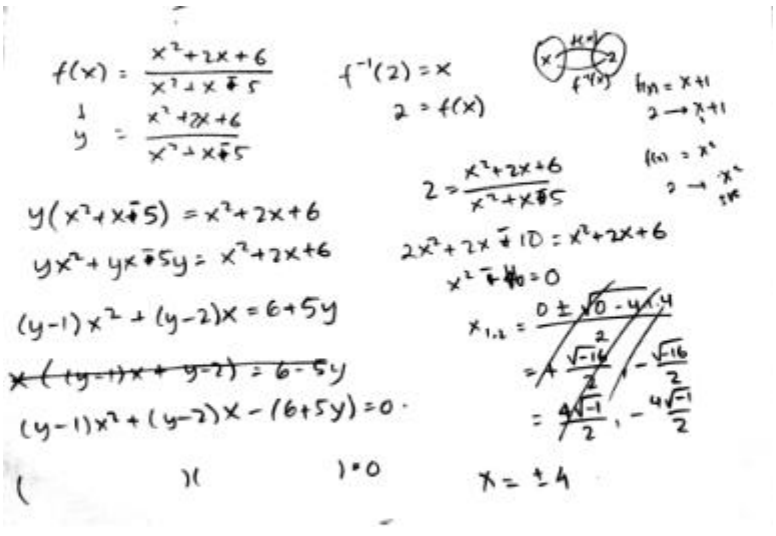

Figure 2. The Work of Ad

(31). He reflected on the way that was done before might produce the same solution, but he was not able to continue the process of solving it.

Table 2 Quotations and coding of the Ad solution for inverse problems

Quote	Behavior	Phase
(1) Reread the problem aloud	Reading problem	Analyzing problem
(2) Asked to find x value of $f^{-1}(2)$	Restating the problem	
(3) I have solved the problem before	Assessing the commonalities of problems	
(4) This is inverse [underlining]	Identifying the main idea of the problem	
(5) I think I should find the inverse, for example, $f(x)$ replaced y	Investigating the problem	
(6) So, y equal to $\frac{x^2+2x+6}{x^2+x-5}$, this can be multiplied	Performing procedural operation	Implementation
(7) I am grouping similar variables		
(8) Divided by x		
(9) Wait, it seems I forget	Checking out the steps he took	
(10) Oh well, this can be factored I think	Making guesses	
(11) Hmmm...[silent for 15 seconds] How to? [writing $(y-1)x^2+(y-2)x=6+5y$]	Experiencing obstacle	
(12) Oh.. it means it can be one and be factored	Identifying new idea	
(13) Hold on... [silent for 15 seconds] What do you think? It looks like I'm stuck here. It's actually not factored	Experiencing obstacles to continue the process	Evaluation
(14) I plan to make x into one side and just replace it with the f^{-1}	Doing reflection	
(15) Here we know the $f(x)$ and find x from $f^{-1}(2)$	Reread the problem	Analyzing problem
(16) The previous problem for $f(x)$ is not like this, and I usually work with this way	Recall the commonalities of the problems	
(17) It seems like this is more difficult than before	Assessing the difficulty of the problem	
(18) Wait a minute, ⋯ I see. I think f^{-1} I can make a connection ⋯	Reexamine the main idea	
(19) If, for example, $f(x)$ is equal to $x+1$, $f^{-1}(2)$ is how much x value that results from 2 means x is equal to 1	Identifying special cases (out of context of the problem)	Formulating the results of the analysis
(20) I try if $f(x)$ is equal to x^2, the x is in the root form		
(21) This means that x results in 2	Summarizing preliminary allegations based on the identification of special cases	
(22) Oh wait, if, for example, $f^{-1}(2)$ is equal to x. It means that 2 is equal to $f(x)$	Comparing the initial guess with the problem at hand	
(23) The inverse of 2 is x, if, for example, you draw x here [as domaint] and 2 here [as range]. This $f(x)$ and This is the inverse. This means for finding x using this way.	Justifying new ideas	

Quote	Behavior	Phase
(24) Ohh..it turns out like this	Assessing the difficulty	
(25) So, 2 is equal to $f(x)$	Finding strategies to solve problems	Implementation
(26) This 2 equals $\frac{x^2+2x+6}{x^2+x-5}$, means $2x^2+2x-10=x^2+2x+6$	Performing procedural operation	
(27) The right side is moved to the left, so $x^2-16=0$		
(28) So, x is 4 and -4	Finding solution	
(29) Wait, can x value be 2. I think this is correct	Justifying solution	Evaluation
(30) If I replace x with 4 and -4 here, the result is 2. Exact		
(31) It's challenging. I didn't expect to have thought this far	Assessing the difficulty	Reflection
(32) Perhaps the previous method would have been used, but it is easier	Reflecting on the process	

3. Finding and Discussion

The research question in this article is how is the process of reversible reasoning occurring? We found factors that cause Ad to reason reversibly on inverse problems; first. Intuitive problem analysis resulted in Ad experiencing obstacles and unable to continue the process of solving; second, the role of metacognitive to build causal relationships by re-analyzing the problem and developing other heuristic strategies. Intuitive problem analysis is seen when Ad compares the problem encountered with previous experience in solving inverse problems using a general strategy, namely swapping the independent (x) and dependent (y) variables from the original function and the solution for the dependent variable is defined as the original function. This is one of the genetic decompositions of the findings of Paoletti et al. (2018) regarding the meaning of the inverse as an act of exchanging x and y and its solution for y. Based on the theory of the anticipation scheme (Ron Tzur, 2011; Simon, Kara, et al., 2016; Simon, Placa, et al., 2016; Tzur, 2007; Tzur & Simon, 2004), intuitive problem analysis results in Ad trying to achieve previous experience based on familiarity so as not constructing an inverse meaning as the reverse of a function process.

In the dual-process theory developed by Leron (Leron & Hazzan, 2009; Leron & Paz, 2014), the analysis of the dominant problem on the intuitive side causes errors, obstacles, and misconceptions when students solve problems. This happened to Ad through the strategy of swapping independent and dependent variables based on previous experience. He experienced obstacles and was unable to continue the settlement process. It does not have an anticipatory scheme to describe the solution to x in y to find the roots of the polynomial equation. Due to experiencing obstacles, there were attempts to reflect by double-checking the completion steps and the strategies used. This is what distinguishes it from other subjects because sometimes it is found that students experience problems and there is no effort to find other alternatives.

Ad reflects back on the problem by looking at the sequence of activities during problem-solving, comparing similar problems based on previous experiences, and assessing the level of difficulty. This is in accordance with Sriraman's opinion (Sriraman, 2015) that mental activity in reflecting includes guessing and examining plausible examples and non-examples, linking with previous experiences, making decisions during and after execution and verification, thinking about similarities in problems, and abstract structural similarities to the problem. Meanwhile, according to Tzur (Ron Tzur, 2011; Tzur, 2007), reflecting on the activity-effect relationship is the assimilation of the problem situation into

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a conception, where assimilation sets goals (resulting in an x value of $f^{-1}(x)$) and invokes a sequence of activities (strategy of exchanging independent and dependent variables) to achieve these goals.

The awareness to reflect by re-analyzing problems analytically and detecting faulty cognition is one aspect of metacognitive thinking (Kim, Park, Moore, & Varma, 2013; Yimer & Ellerton, 2010). This can lead to a new conception to build a causal relationship between function and inverse problems. One of Ad's heuristic strategies is to reduce the complexity of the problem marked by highlighting the meaning of the inverse and identifying special cases outside the context of the problem ($f(x) = x + 1$ dan $f(x) = x^2$), and analyzing cases. by expressing "if $f^{-1}(2)$ of this f [refers to $f(x) = x + 1$], then x is equal to 1" repeatedly. By analogy, he found that the problem was identical to finding "how many x gives 2" and proved it by visualizing the problem in a venn diagram. This is in line with the opinion of Haciomerouglu (Haciomeroglu, 2007; Haciomeroglu et al., 2009, 2010), Flanders (Flanders, 2014), and Lubin et al. (Lubin et al., 2015) that interpreting problems verbally and visualizing problems is sometimes needed to find problem situations.

The awareness to reflect by re-analyzing problems analytically and detecting faulty cognition is one aspect of metacognitive thinking (Kim, Park, Moore, & Varma, 2013; Yimer & Ellerton, 2010). This can lead to a new conception to build a causal relationship between function and inverse problems. One of Ad's heuristic strategies is to reduce the complexity of the problem marked by highlighting the meaning of the inverse and identifying special cases outside the context of the problem (f (x) = x + 1 and f (x) = x ^ 2), and analyzing cases. by expressing "if f ^ (- 1) (2) of this f [refers to f (x) = x + 1], then x is equal to 1" repeatedly. By analogy, he found that the problem was identical to finding "how many x gives 2" and proved it by visualizing the problem in a venn diagram. This is in line with the opinion of Haciomerouglu (Haciomeroglu, 2007; Haciomeroglu et al., 2009, 2010), Flanders (Flanders, 2014), and Lubin et al. (Lubin et al., 2015) that interpreting problems verbally and visualizing problems is sometimes needed to find problem situations.

The results of case identification are used to compare with the problem at hand (found x from $f^{-1}(2)$) and found a solution to the problem. In this case, Ad interprets inverse as the opposite of a functional process, which is one of the genetic decompositions developed by Paoletti et al. (Paoletti et al., 2018). Because Krutetskii (Krutetskii, 1976) interprets reversible reasoning as reconstructing the direction of thinking, we conclude that Ad experienced a change in point of view from the act of swapping the independent and dependent variables switching to the act of interpreting the inverse as the opposite of the function process. The existence of obstacles due to the intuitive analysis of the problem and metacognitive thinking is the reason why Ad does reversible reasoning. This might happen to other students.

Based on the theory of anticipatory schemes developed by Steffe & Olive (Lesllie P. Steffe & Olive, 2009), Hackenberg (Hackenberg, 2010) and Simon & Tzur (Ron Tzur, 2011; Simon, Kara, et al., 2016; Simon, Placa, et al., 2016; Tzur, 2007; Tzur & Simon, 2004), we found that before using the results of the schema (inverse function) to generate causes (inverse as the inverse of a functional process), students performed various mental activities through reflection by restating experiences beforehand and developed a heuristic strategy. Generating an inverse meaning as the opposite of a functional process is Ad's attempt to build a causal relationship into new experiences (identifying cases outside of the problem), and separating causes in order to obtain results by rearranging the problem (comparing the results of identification outside the problem). In addition, if viewed from the dual-process theory (System 1 and System 2) (Leron & Hazzan, 2009; Leron & Paz, 2014), what Ad shows is the progress of thinking by providing an analytic role as a monitor and criticism to correct or replace intuitive analysis (change from System 1 to System 2).

4. Conclusion

This study investigates a special case in Ad about the process of reversible reasoning in inverse problems. The process of reversible reasoning is that Ad experiences obstacles due to intuitive problem analysis and is unable to continue the solving process. Furthermore, there is metacognitive thinking by realizing that it is necessary to re-analyze the problem analytically. We note the change in point of view for the meaning of the inverse, namely: first, Ad does not have an anticipatory scheme in describing the solution for x and y to find the root of the polynomial equation when interpreting the inverse as the act of swapping the independent and dependent variables; and second, Ad develops another heuristic strategy by reducing the complexity of the problem, identifying special cases outside the context of the problem. The results of the identification of these cases are used to compare with the problem at hand to find the inverse meaning as the opposite of the function process.

Our findings develop the results of previous research that reversible reasoning is not only sensitive to the numerical nature of the problem parameters, but can be constructed based on a cause-and-effect relationship (Ramful, 2009, 2014, 2015), and there needs to be a trigger to find alternative pathways in reversing the problem situation (Flanders, 2014; Haciomeroglu et al., 2009, 2010). In addition, we agree that the anticipation scheme is a prerequisite for reversible reasoning (Hackenberg, 2010; Ron Tzur, 2011; Simon, Kara, et al., 2016; Simon, Placa, et al., 2016; Tzur & Simon, 2004). Finally, the reversible reasoning that Ad shows is one of the types we encountered in this study. We use the term "refraction", which is a change in perspective that occurs due to obstacles, the inability to continue the problem-solving process, and the presence of metacognitive thinking to develop heuristic strategies.

We note several important things for future research, namely we suspect that there are still several types of reversible reasoning, especially for students who do not directly experience obstacles. There is a possibility that they may do reversible reasoning in solving problems. Although we focus on the inverse problem, reversible reasoning can be investigated for other concepts, such as calculus, that is, finding the original function based on the derivative of a known function or sketching a graph of the function based on the graph of the derived function and establish a relationship between differentiation and integration processes.

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