

Partially Ordered Group of the 2×2 Symmetric Matrices

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Abstract

This paper present results partially ordered group on the set of 2×2 symmetric matrix through its positive cones. There are two positive cones that is the subset of the symmetric matrix. The first cone is constructed by defining positive matrix as the matrix that each entry of the matrix is positive. The second cone is constructed by using the characterization of the matrix.

Keywords: Ordered Group, Symmetric Matrix, Positive Cone.

Abstrak

Artikel ini menyajikan grup terurut parsial dari himpunan matriks simetri berukuran 2×2 yang dikonstruksi melalui cone positifnya. Terdapat dua cone positif yang merupakan subhimpunan pada matriks simetri. Pertama, cone positif dibentuk dengan mendefinisikan matriks positif sebagai matriks yang setiap entrinya positif. Kedua, cone positif dibentuk melalui karakteristik karakteristik dari matriks simetri.

Kata kunci: Grup Terurut, Matriks Simetri, Positif Cone.

1. INTRODUCTION

In this paper, we consider the set of 2×2 matrix over real number

$$M_2(R) = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \mid a_1, a_2, a_3, a_4 \in R \right\},$$

which is group under addition operation matrix. The matrix over real number $M_2(R)$ is said to be partially ordered group if $A \leq B$ for any two matrices A and B that satisfies $a_i \leq b_i$ for all $i = 1, 2, 3, 4$.

Define the subset $S_2(R)$ of group $M_2(R)$

$$S_2(R) = \left\{ \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix} \mid a_1, a_2, b \in R \right\}$$

with the addition operation. It can be shown that the set $S_2(R)$ is also group.

Let two elements A and B in symmetric matrix $S_2(R)$. Now we define the order of the two elements by $A \geq B$ if and only if for all x is an element of R^2 satisfies $\langle x, (A-B)x \rangle = x^T (A-B)x \geq 0$ [1]. By the defined order, the 2×2 symmetric matrix $S_2(R)$ with positive cone [2]

$$P = \left\{ A = \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix} \in S_2(R) \mid \langle x, Ax \rangle \geq 0, \forall x \in R^2 \right\},$$

constructs partially ordered group.

2. METHOD

2.1. Group

Let G is a nonempty set with operation “ $*$ ”. The set G is a group if [1]:

- (1) For any two elements in G , a and b , satisfies $a * b$.
- (2) For any three elements in G , a, b , and c , satisfies $(a * b) * c = a * (b * c)$.
- (3) There exists an identity element e in G and for all a element in G satisfies $e * a = a * e$.
- (4) For all element a in G , there exists an inverse of a , that is a^{-1} , satisfies $a^{-1} * a = a * a^{-1}$.

The real numbers R is one of a group under addition operation. But it is not group under multiplication operation since there is no inverse of the number zero in R .

2.2. Poset

Let P is a nonempty set with Binary relation “ \leq ” is said to be partially ordered set, called poset, if the following properties are satisfied for all x, y, z elements in P [3]:

- (1) Reflexive: $x \leq x$.
- (2) Antisymmetric: $x \leq y$ and $y \leq x$ implies $x = y$.
- (3) Transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$.

Moreover, a set P is said to be totally ordered set if it is partially ordered set that satisfies $x \leq y$ or $y \leq x$ for all elements x and y in P . For instance, the real number set R with Binary relation “ \leq ”. It is the partially ordered set since for any two real numbers x and y , $x \leq y$ or $y \leq x$. It is always be ordered. Then, let us define the Cartesian product of the real number set R as follow:

$$R \times R = \{(x, y) \mid x, y \in R\}.$$

By defined the order as

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2,$$

It is also partially ordered set but it is not totally ordered set since there exists $(1,4)$ and $(2,3)$ on $R \times R$ that is not satisfies whether $(1,4) \leq (2,3)$ or $(1,4) \geq (2,3)$.

2.3. Inner Product

Let x and y are two elements of n dimensional R , R^n . The standard inner product of R is a function

$$\langle \cdot, \cdot \rangle : R^n \times R^n \rightarrow R^n,$$

that is defined by $\langle x, y \rangle = y^T x$ with y^T is a transpose of y . The properties of the standard inner product for any elements x, y , and z in R^n and the scalar α in R as follows [4] [5]:

- (1) Symmetric: $\langle x, y \rangle = \langle y, x \rangle$.
- (2) Positive definite: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.
- (3) Linearity of the addition: $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.
- (4) Linearity of multiplication: $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

Recall the set of 2×2 symmetric matrix over R ,

$$S_2(R) = \left\{ \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix} \mid a_1, a_2, b \in R \right\}.$$

Let any element $A \in S_2(R)$. By the inner product of matrix defined

$$\langle Ax, x \rangle = x^T Ax,$$

Then

$$\langle x, Ax \rangle = (Ax)^T x = x^T A^T x = x^T Ax = \langle Ax, x \rangle.$$

Since the symmetric matrix has the property $A^T = A$. The following theorem gives the characteristics of symmetric matrix.

Theorem 1

Let $A \in S_n(\square^n)$ symmetric matrix, then the inner product $\langle Ax, x \rangle = x^T Ax$ is real number for all $x \in \square^n$ [6].

2.4. Partially Ordered Group

A Partially ordered set G which is also a group is said to be partially ordered group if the order of G is preserved under addition that is if $x \leq y$ then $a + x \leq a + y$ and $x + a \leq y + a$ for any elements x, y in G and the scalar a in R . Define the positive cone G^+ by all positive elements of G , $G^+ = \{g \in G \mid g \geq 0\}$, and the negative cone G^- by all negative elements of G , $G^- = \{g \in G \mid -g \in G^+\}$. Any group can be constructed as partially ordered group with the positive cone.

Theorem 2

Let P is the subset of group G under addition that satisfies two properties:

- (1) For any two element a and b of P , $a + b$ is element of P .
- (2) The intersection of the set P and its negative, $-P$, is identity element 0 .

Then the set G can be constructed as partially ordered group by defined

$$a \geq b \text{ if and only if } a - b \in P$$

Moreover, this results the group G to be partially ordered group with the positive cone P . We denote the partially ordered group with its cone by (G, P) .

3. RESULT AND DISCUSSION

Recall again the set of 2×2 symmetric matrices

$$S_2(R) = \left\{ \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix} \mid a_1, a_2, b \in R \right\}$$

that is group under matrix addition operation. The following present two positive cones constructed by the subset of $S_2(R)$.

Lemma 1

Let P_0 be a subset of $S_2(R)$ defined by

$$P_0 = \left\{ A = \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix} \in S_2(R) \mid a_1, a_2, b \geq 0, \forall x \in R^2 \right\}$$

The set $S_2(R)$ with P_0 is partial ordered set with order

$$A \geq B \text{ if and only if } A - B \in P_0.$$

Proof:

It will be shown that the set $S_2(R)$ with the P_0 is partially ordered.

1. For any two elements $A = \begin{pmatrix} u_1 & w_1 \\ w_1 & v_1 \end{pmatrix}$ and $B = \begin{pmatrix} u_2 & w_2 \\ w_2 & u_2 \end{pmatrix}$ of P_0 ,

$$A + B = \begin{pmatrix} u_1 & w_1 \\ w_1 & v_1 \end{pmatrix} + \begin{pmatrix} u_2 & w_2 \\ w_2 & v_2 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 & w_1 + w_2 \\ w_1 + w_2 & v_1 + v_2 \end{pmatrix} \in P_0$$

Since $A + B \in S_2(R)$ with $u_1 + u_2, v_1 + v_2, w_1 + w_2 \geq 0$.

2. If $A = \begin{pmatrix} u & w \\ w & v \end{pmatrix}$ is an element of the intersection of P_0 and its negative $-P_0$ implies

$A = \begin{pmatrix} u & w \\ w & v \end{pmatrix} \in P_0$ and $-A = \begin{pmatrix} -u & -w \\ -w & -v \end{pmatrix} \in P_0$. By definition, u, v , and w are elements of R and are positive. Thus, $-u, -v$, and $-w$ are negative. In this case, the elements u, v , and w of R that satisfy the condition is zero.

Based on the Theorem 2, by the order

$$A \geq B \text{ if and only if } A - B \in P_0,$$

For all A and B are elements of the 2×2 symmetric matrix $S_2(R)$. The matrix $S_2(R)$ can be constructed as partially ordered set. Moreover, Group $S_2(R)$ become partially ordered group and its positive cone is P_0 , denoted by $(S_2(R), P)$.

Lemma 2

Let P_1 be a subset of $S_2(R)$ defined by [7]:

$$P_1 = \left\{ \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix} \in S_2(R) \mid \langle x, Ax \rangle \geq 0, \forall x \in R^2 \right\}.$$

The set $S_2(R)$ with P_1 is partial ordered set with order:

$$A \geq B \text{ if and only if } A - B \in P_1.$$

Proof:

It will be shown that the set $S_2(R)$ with the P_1 construct partial order.

1. For any two elements $A = \begin{pmatrix} u_1 & w_1 \\ w_1 & v_1 \end{pmatrix}$ and $B = \begin{pmatrix} u_2 & w_2 \\ w_2 & v_2 \end{pmatrix}$ of P_1 , then A and B are elements of $S_2(R)$ that satisfies $\langle x, Ax \rangle \geq 0$ and $\langle x, Bx \rangle \geq 0$ for all $x \in R^2$. By the properties of $S_2(R)$ as group, $A + B$ is element of $S_2(R)$. By the properties of the inner product on $S_2(R)$,

$$\langle x, (A + B)x \rangle = \langle x, Ax \rangle + \langle x, Bx \rangle \geq 0.$$

Therefore, $A + B \in P_1$.

2. Let A is an element of the intersection of P_1 and its negative. By definition, A is an element of P_1 and A is an element of $-P_1$. Thus $\langle x, Ax \rangle \geq 0$ and $-\langle x, Ax \rangle \geq 0$. Since $\langle x, Ax \rangle = x^T Ax$ is real number, then $\langle x, Ax \rangle = 0$ and non zero $x \in R^2$ satisfies $A = 0$.

Based on the Theorem 2, by the order

$$A \geq B \text{ if and only if } A - B \in P_1$$

For all A and B are elements of $S_2(R)$. The matrix $S_2(R)$ can be constructed as partially ordered. Moreover, the group $S(R)$ with its positive cone P_1 become partially ordered group, and is written as $(S_2(R), P_1)$.

4. CONCLUSION

Based on the explanation, it has been shown that there exists two different group orders of the 2×2 symmetric matrices $S_2(R)$. The two orders defined through positive cone P_0 and P_1 construct $(S_2(R), P_0)$ and $(S_2(R), P_1)$, respectively, to be partially ordered groups.

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