Buletin Teknik Elektro dan Informatika
(Bulletin of Electrical Engineering and Informatics)
Vol. 3, No. 3, September 2014, pp. 181~186
ISSN: 2089-3191

A Time-Varying System - Missile Dynamics

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Abstract

Most of the control theory is developed around time-invariant systems where the state matrix A consists of scalars which are not functions of time. However, many physical systems are naturally modeled with the elements of the state matrix A depending on time. One example is the dynamics of a missile. Time-varying systems also arise when non-linear systems are linearized about a trajectory. In this work, the state-transition matrix is studied for time-varying systems in order to reach a general solution. The computational effort is significantly more complicated than the time-invariant case. There are many different methods in the literature for finding the state-transition matrix and one of them is adopted. Finally a case study of Missile Dynamics will be analyzed and simulated using MATLAB.

Keyword:

1. Introduction

In this work systems of the form $\dot{x}(t)$ = A(t)x(t) + B(t)u(t); $x(t_0) = x_0$ will be considered where A(t) and B(t) are continuous functions of t and u(t) is a piecewise continuous function of t. X(t) is the system state vector and u(t) is the input vector. Several people dealt with these systems and for more information the reader may consult references [1]-[3],[6]. The purpose of this research is to understand the link between mathematical models of the above form as applied to missile dynamics developed in reference [4]. The use of MATLAB is necessary to see the behavior of time-varying systems as the computing requirements are very complex. The reader may obtain more information about MATLAB, which stands for MATrix LABoratory in reference [5].

In the first part of this work some theoretical results are analyzed as well as a good computational method is adopted, see also [6], for finding the transition matrix $\Phi(t, t_0)$ of time-varying systems. The motivation is based on a system of missile dynamics whose transition (dynamics) matrix A(t) depends on time. As explained in [4] the matrix $\Phi(t, t_0)$ relates the state at time t to the state at time t_0. It defines how the state $x(t_0)$ transitions into state x(t). The second part will consist of the case study(missile dynamics), the analytical solution and the simulation part.

2. Theory Of Time-Varying System

Consider the general linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

The general solution is given by with matrix dimensions as A: n x n, B: n x p, C: r x n and D: r x p.

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau$$

Received April 2, 2014; Revised June 13, 2014; Accepted June 28, 2014

where $\Phi(t, t_0)$ is the state transition matrix. In the time-invariant case this matrix is a simple exponential. In the time-varying case, however, there is no such a simple expression, in fact, $\Phi(t, t_0)$ varies from system to system. Furthermore, knowledge of $\Phi(t, 0)$ is not adequate information for finding $\Phi(t, t_0)$. The state transition matrix must always satisfy the following relationships:

$$\frac{\partial \phi(t, t_0)}{\partial t} = A(t)\phi(t, t_0)$$

$$\phi(\tau, \tau) = I$$

where I is the identity matrix.

In addition $\Phi(t, t_0)$ has the following properties: For proofs see [3],[6].

a. $\phi(t_2, t_1)\phi(t_1, t_0) = \phi(t_2, t_0)$

b. $\phi^{-1}(t, \tau) = \phi(\tau, t)$

c. $\phi^{-1}(t, \tau)\phi(t, \tau) = I$

d. $\dfrac{d\phi(t, t_0)}{dt} = A(t)\phi(t, t_0)$

When the system is time invariant, the state transition matrix is given by

$\phi(t, t_0) = e^{A(t-t_0)}$ which satisfies the above properties.

A useful method in finding the state transition matrix for time-varying systems is the following: The reader may consult [6] for more information.

If A(t) can be decomposed as the following sum,

$$A(t) = \sum_{i=1}^{n} M_i f_i(t)$$

where M_i is a constant matrix (that it commutes) such that $M_i M_j = M_j M_i$, and f_i is a single-valued function, then the state-transition matrix can be given as:

$$\phi(t, \tau) = \prod_{i=1}^{n} e^{M_i \int_\tau^t f_i(\theta) d\theta}$$

A very good example is taken from [7].

Let the system matrix A for the time-varying system be

$$A = \begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}$$

This matrix can be decomposed as follows:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} t + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Where f1(t) = t, and f2(t) = 1. Using the formula described above gives us:

$$\phi(t,\tau) = e^{M_1 \int_\tau^t \theta d\theta} e^{M_2 \int_\tau^t d\theta}$$

Solving the two integrations gives us:

$$\phi(t,\tau) = e^{\frac{1}{2}\begin{bmatrix} (t^2-\tau^2) & 0 \\ 0 & (t^2-\tau^2) \end{bmatrix}} e^{\begin{bmatrix} 0 & t-\tau \\ -t+\tau & 0 \end{bmatrix}}$$

The first term is a diagonal matrix, and the solution to that matrix function is all the individual elements of the matrix raised as an exponent of e. The second term can be decomposed as:

$$e^{\begin{bmatrix} 0 & t-\tau \\ -t+\tau & 0 \end{bmatrix}} = e^{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(t-\tau)} = \begin{bmatrix} cos(t-\tau) & sin(t-\tau) \\ -sin(t-\tau) & cos(t-\tau) \end{bmatrix}$$

The final solution is given as:

$$\phi(t,\tau) = \begin{bmatrix} e^{\frac{1}{2}(t^2-\tau^2)} & 0 \\ 0 & e^{\frac{1}{2}(t^2-\tau^2)} \end{bmatrix} \begin{bmatrix} cos(t-\tau) & sin(t-\tau) \\ -sin(t-\tau) & cos(t-\tau) \end{bmatrix}$$

$$= \begin{bmatrix} e^{\frac{1}{2}(t^2-\tau^2)} cos(t-\tau) & e^{\frac{1}{2}(t^2-\tau^2)} sin(t-\tau) \\ -e^{\frac{1}{2}(t^2-\tau^2)} sin(t-\tau) & e^{\frac{1}{2}(t^2-\tau^2)} cos(t-\tau) \end{bmatrix}$$

It is worth to notice that the constant matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ whose matrix exponential is $\begin{bmatrix} \cos(t) & \sin(t) \\ -sint(t) & \cos(t) \end{bmatrix}$. The reader may consult [6] for more details and proofs.

3. Case Study – Missile Dynamics

Figure 1 [4] shows the geometry of a missile and target both confined to move in a plane. The missile moves at constant speed V_M and the target moves in a straight line at constant velocity V_T. As shown in the figure the direction of the velocity vector can be controlled by the use of an acceleration α which is assumed to be perpendicular to the relative velocity vector $\mathbf{V} = \mathbf{V_M} - \mathbf{V_T}$. The various parameters depicted in the figure are: r is the range to the target, λ is the inertial line-of-sight angle, σ is the angle subtended at the missile by the velocity vector and the line of sight, γ is the flight path angle and α is the applied acceleration.

Let z be the projected miss distance (distance of closest approach of the missile to the target) under the assumption that the missile continues in a straight line without any further acceleration.

Then, z = rsin(σ) and using the dynamics of relative motion, it can be shown that $\frac{dz}{dt} = \frac{rcos(\sigma)}{V}\alpha$. Furthermore, assuming that σ is a small angle, then $\frac{dr}{dt} \approx -V$. Thus, r(t) = r_0 – Vt, then $\frac{r}{V} \approx T_0$ – t = $\bar{T}$ which is frequently called "time-to-go" where $T_0 = \frac{r_0}{V}$.

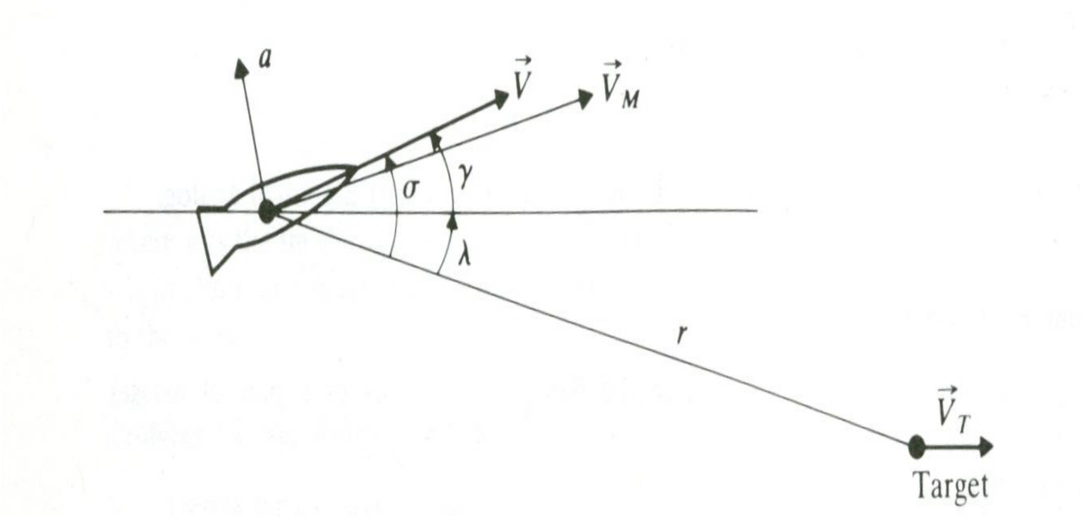

Figure 1. Missile Dynamics Guidance Set-up [4]

With all these assumptions, the following state – space representation of the approximate missile dynamics is

$$\begin{bmatrix} \dot{\lambda} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{V(T_0 - t)^2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ T_0 - t \end{bmatrix} \alpha \text{ with } A(t) = \begin{bmatrix} 0 & \frac{1}{V(T_0 - t)^2} \\ 0 & 0 \end{bmatrix} \text{ and}$$

$$B(t) = \begin{bmatrix} 0 \\ T_0 - t \end{bmatrix}.$$

As it can be observed the A matrix is time-varying, so for the solution the transition matrix Φ(t) is needed using the method as described in the theory section. So, using the same method as in the previous section, it can be obtained.
This A(t) matrix can be decomposed as follows:

$$A(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{V(T_0 - t)^2} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Where f1(t) = $\frac{1}{V(T_0-t)^2}$, and f2(t) = 1. Using the formula

$$\Phi(t,\tau) = e^{M_1 \int_\tau^t \frac{1}{V(T_0-\theta)^2} d\theta} e^{M_2 \int_\tau^t d\theta}$$

Solving the two integrations gives us:

$$\Phi(t,\tau) = e^{\begin{bmatrix} 0 & (\frac{1}{T_0-t} - \frac{1}{T_0-\tau})\frac{1}{V} \\ 0 & 0 \end{bmatrix}} e^{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

The second term gives the identity matrix = **I**, a nice property of the matrix exponential. The first term can be decomposed as:

$$e^{\begin{bmatrix}0 & (\frac{1}{T_0-t}-\frac{1}{T_0-\tau})\frac{1}{V}\\0 & 0\end{bmatrix}} = e^{\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}(\frac{1}{T_0-t}-\frac{1}{T_0-\tau})\frac{1}{V}}$$

Now using one of the so many methods $e^{\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}}$ can be evaluated to be $\begin{bmatrix}1 & t\\0 & 1\end{bmatrix}$.

So, $e^{\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}(\frac{1}{T_0-t}-\frac{1}{T_0-\tau})\frac{1}{V}} = \begin{bmatrix}1 & (\frac{1}{T_0-t}-\frac{1}{T_0-\tau})\frac{1}{V}\\0 & 1\end{bmatrix}$

Finally, $\Phi(t,\tau) = \begin{bmatrix}1 & (\frac{1}{T_0-t} - \frac{1}{T_0-\tau})\frac{1}{V}\\0 & 1\end{bmatrix} \mathrm{I} = \begin{bmatrix}1 & (\frac{1}{T_0-t} - \frac{1}{T_0-\tau})\frac{1}{V}\\0 & 1\end{bmatrix}$.

Finding the transition matrix, a closed form solution to the problem can be found using

$$\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau$$

The simulation MATLAB programs are shown in Figure 2 by setting $x_1 = \lambda$ and $x_2 = z$ with V = 375 m/s, u = α = 1250 m/s^2 and T_0 = 10 s for a hypothetical highly-maneuverable missile.

```
%missile guidance time-varying system
function [Dx] = timevaryingsystem1(t,x)
V=375;To=10; %V is velocity of missile in meters per second %To is initial time in seconds
u=1250;%acceleration normal to the missile relative velocity vector in meters per second square
Dx = [0*x(1)+(1/(V*(To - t).^2))*x(2)+0*u;0*x(1)+0*x(2)+(To-t)*u];
```

```
%this is the main time_varying system program calling the function
%timevaryingsystem1
t_values=linspace(0,10,101);%time vector
initial_cond=[0;0];
[tv,Yv]=ode45('timevaryingsystem1',t_values,initial_cond);
plot(tv,Yv(:,1),'+',tv,Yv(:,2),'--')
legend('y1','y2')
xlabel('time in seconds')
ylabel('y1(angle-radians), y2(distance-meters')
title('Missile Guidance')
```

Figure 2. MATLAB programs used for the simulation

The simulation results are

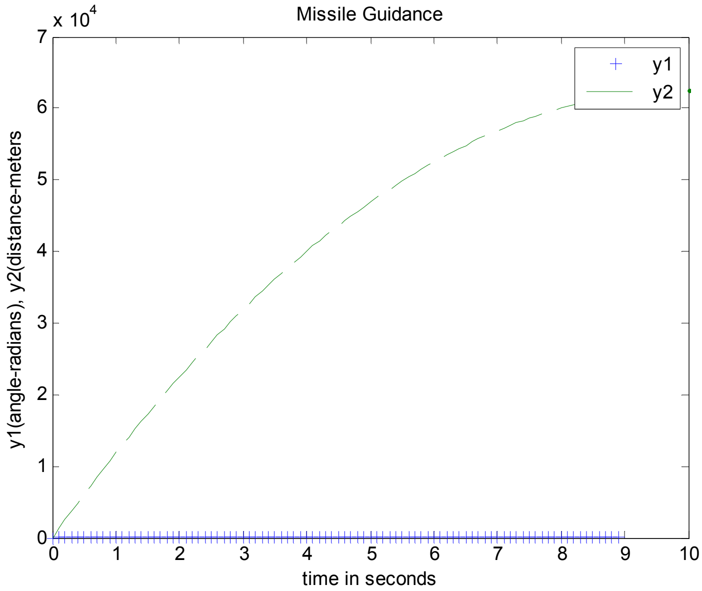

Figure 3. Simulation of Missile Guidance Dynamics

The simulation clearly demonstrates the approximation of a small σ (sigma) angle and the distance of closest approach travelled by the missile of about 62500 meters.

4. Conclusion

In this research endeavor the necessity of finding the transition matric Φ(t ,τ) of a time-varying system is demonstrated. Obtaining this matrix, a general solution (close form) can be found. Reference [6] is an excellent source for the reader to consult for more details and proves. The missile guidance example is a very important one even if it is simplified. The calculation of the transition matrix for time-varying systems is considerably more tedious than time-invariant systems. The simulation results were as expected.

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