

Codes Detecting and Correcting Solid Burst Errors

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Abstract: This paper studies linear codes capable of detecting and correcting solid burst error of length b or less. The lower and upper bounds on the number of parity-check digits required for such codes are obtained. Illustrations of codes for detecting as well as correcting such errors are provided.

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1. INTRODUCTION

Investigations in coding theory have been made in several directions but one of the most important directions has been the detection and correction of errors. It began with Hamming codes [9] for single errors, Golay codes ([5], [6]) for double and triple random errors and thereafter BCH codes ([7], [8], [10]) were studied for multiple error correction. There is a long history towards the growth of the subject and many of the codes developed have found applications in numerous areas of practical interest. One of the areas of practical importance in which a parallel growth of the subject took place is that of burst error detecting and correcting codes. It has been observed that in many communication channels, burst errors

occur more frequently than random errors. A burst of length b may be defined as follows:

Definition 1. *A burst of length b is a vector whose only non-zero components are among some b consecutive components, the first and the last of which is non zero.*

When in a burst of length b , all the b components, in which the non zero components are confined, are non zero i.e., all the digits among the b components are in error, such type of burst is known as solid burst. Solid burst are prevalent in channels viz. semiconductor memory data[11], supercomputer storage system[2]. A solid burst may be defined as follows:

Definition 2. *A solid burst of length b is a vector with non zero entries in some b consecutive positions and zero elsewhere.*

Schillinger[14] developed codes that correct solid burst error. Shiva and Cheng[16] produced a paper for correcting multiple solid burst error of length b in binary code with a very simple decoding scheme. Among many, some of the good research on solid burst can be mentioned such as Bossen[3], Sharma and Dass[15], Etzion[4], Argyrides, Reviriego, Pradhan and Maestro[1].

It is very clear that the nature of error differs from channel to channel depending upon the behaviour of channels or the kind of errors which occur during the process of transmission. There is a need to deal with many types of error patterns and accordingly codes are to be constructed to combat such error patterns. Though many works have been done on solid burst of length b or less, the bounds on the number of parity checks for linear codes over $GF(q)$ detecting and correcting such errors are not obtained properly. In this paper, the author presents the bounds for

such codes.

The rest of the paper is organized into two sections. In the second section, the lower and upper bounds on the number of parity check digits of linear codes that detect any solid burst of length b or less are obtained. The section 3 presents the similar bounds for codes correcting such errors.

In what follows a linear code will be considered as a subspace of the space of all n -tuples over $\text{GF}(q)$. The distance between two vectors shall be considered in the Hamming sense.

2. CODES DETECTING SOLID BURST ERRORS

We consider linear codes over $\text{GF}(q)$ that are capable of detecting any solid burst error of length b or less. Clearly, the patterns to be detected should not be code words. In other words we consider codes that have no solid burst error of length b or less as a code word. Firstly, we obtain a lower bound over the number of parity-check digits required for such a code. The proof is based on the technique used in theorem 4.13, Peterson and Weldon [12].

Theorem 1. *Any (n, k) linear code over $\text{GF}(q)$ that detects any solid burst of length b or less must have at least $\log_q(1 + b)$ parity-check digits.*

Proof. The result will be proved on the basis that no detectable error vector can be a code word.

Let V be an (n, k) linear code over $\text{GF}(q)$. Consider a set X of all those vectors such that the some fixed non-zero components are in b or less consecutive positions starting from the first position.

We claim that no two vectors of the set X can belong to the same coset of the standard array; else a code word shall be expressible as a sum or difference of two error vectors.

Assume on the contrary that there is a pair, say x_1, x_2 in X belonging to the same coset of the standard array. Their difference viz. $x_1 - x_2$ must be a code vector. But $x_1 - x_2$ is a vector all of whose non zero components are confined consecutively to b or less components i.e., $x_1 - x_2$ is a solid burst of length b or less, which is a contradiction. Thus all the vectors in X must belong to distinct cosets of the standard array. The number of such vectors over $GF(q)$, including the vector of all zero, is clearly

$$(1) \qquad 1 + b$$

The theorem follows since there must be at least this number of cosets. ■

In the following theorem, an upper bound on the number of check digits required for the construction of a linear code considered in theorem 1 is provided . This bound assures the existence of a linear code that can detect all solid burst error of length b or less. The proof is based on the well known technique used in Varshomov-Gilbert Sacks bound by constructing a parity check matrix for such a code (refer Sacks[13], also theorem 4.7 Peterson and Weldon [12]).

Theorem 2. *There exists an (n, k) linear code over $GF(q)$ that has no solid burst of length b or less as a code word provided that,*

$$n - k > \log_q \left\{ \sum_{i=0}^{b-1} (q - 1)^i \right\}.$$

Proof. The existence of such a code will be shown by constructing an appropriate $(n - k) \times n$ parity-check matrix H . The requisite parity-check matrix H shall be

constructed as follows:

Select any non-zero $(n - k)$ -tuples as the first $j - 1$ columns h_1, h_2, \dots, h_{j-1} appropriately, we lay down the condition to add j^{th} column h_j such that h_j should not be a linear sum of immediately preceding consecutive upto $b - 1$ columns. In other words

$$(2) \quad h_j \neq u_{j-1}h_{j-1} + u_{j-2}h_{j-2} + \dots + u_{j-s+2}h_{j-s+2} + u_{j-s+1}h_{j-s+1}$$

where $s \leq b$, $j \geq s$, the coefficients $u_i \in GF(q)$ are non zero.

This condition ensures that no solid burst of length b or less will be a code word which thereby means that the code shall be able to detect solid bursts of length b or less.

The number of ways in which the coefficients u_i can be selected, including the vector of all zeros, is

$$(3) \quad 1 + \sum_{i=1}^{b-1} (q-1)^i.$$

At worst, all these linear combinations might yield a distinct sum.

Therefore a column h_j can be added to H provided that

$$q^{n-k} > 1 + \sum_{i=1}^{b-1} (q-1)^i.$$

or,

$$n - k > \log_q \left\{ \sum_{i=0}^{b-1} (q-1)^i \right\}. \quad \blacksquare$$

Example 1. Consider a $(4, 2)$ binary code with the 2×4 matrix H which has been constructed by the synthesis procedure given in the proof of theorem 2 by taking $b = 3, n = 4$.

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The null space of this matrix can be used to detect all solid bursts of length 3 or less. It may be verified from error pattern-syndromes table 1 that the syndromes of all solid bursts of length 3 or less are non zero.

Table 1

Error patterns	Syndromes
Solid bursts of length 1	
1000	10
0100	01
0010	10
0001	01
Solid bursts of length 2	
1100	11
0110	11
0011	11
Solid bursts of length 3	
1110	01
0111	10

3. CODES CORRECTING SOLID BURST ERRORS

Out of the two results obtained in this section, the first result gives a lower bound on the number of check digits required for the existence of a linear code over $GF(q)$ that corrects all solid bursts of length b or less. The second result gives an upper bound on the number of check digits which ensures the existence of such a code. The proof of the first result is based on the technique used in theorem 4.16, Peterson and Weldon[12]. The proof of the second result is based on the same well known technique used in Varshomov-Gilbert Sacks bound by constructing a parity check matrix for such a code (refer Sacks[13], also theorem 4.7 Peterson and Weldon [12]).

Theorem 3. *An (n,k) linear code over $GF(q)$ that corrects all solid bursts of length b or less must have at least $\log_q \left\{ 1 + \sum_{i=1}^b (n-i+1)(q-1)^i \right\}$ parity check digits.*

Proof. The proof is based on counting the number of correctable error vectors and comparing it with the available number of cosets.

We have,

the number of solid bursts of length 1 = $n(q - 1)$

the number of solid bursts of length 2 = $(n - 1)(q - 1)^2$

the number of solid bursts of length 3 = $(n - 2)(q - 1)^3$

.....

.....

the number of solid bursts of length $b = (n - b + 1)(q - 1)^b$

So, the total number of correctable error vectors including the vector of all zeros is

$$1 + \sum_{i=1}^b (n - i + 1)(q - 1)^i$$

For correction, all these vectors must belong to different cosets. The total number of cosets available is q^{n-k} . Therefore we must have

$$q^{n-k} \geq 1 + \sum_{i=1}^b (n - i + 1)(q - 1)^i$$

Or,

$$n - k \geq \log_q \left\{ 1 + \sum_{i=1}^b (n - i + 1)(q - 1)^i \right\}. \quad \blacksquare$$

Now what follows is an upper bound on the number of check digits required for the construction of a linear code discussed in theorem 3. This bound assures the existence of a linear code that can correct all solid burst error of length b or less.

Theorem 4. *There shall always exist an (n, k) linear code over $GF(q)$ that corrects all bursts of length b or less ($n > 2b$) provided that*

$$(4) \quad q^{n-k} > 1 + \sum_{i=1}^b \sum_{l=1}^b (n - l - i + 1)(q - 1)^{i+l-1}$$

Proof. The existence of such a code will be proved by constructing an $(n - k) \times n$ parity check matrix H for the desired code as follows.

Select any non zero $(n - k)$ tuple as the first column h_1 of the matrix H . After having selected the first $j - 1$ columns h_1, h_2, \dots, h_{j-1} appropriately, we lay down the condition to add j^{th} column h_j as follows:

h_j should not be a linear sum of immediately preceding upto $b - 1$ consecutive columns $h_{j-1}, h_{j-2}, \dots, h_{j-b+1}$, together with any b or fewer consecutive columns from amongst the first $j - b$ columns h_1, h_2, \dots, h_{j-b} i.e.

$$h_j \neq (u_{j-1}h_{j-1} + u_{j-2}h_{j-2} + \dots + u_{j-s}h_{j-s})$$

$$(5) \quad + (v_i h_i + v_{i+1} h_{i+1} + \dots + v_{i+s'-1} h_{i+s'-1})$$

where $u_i, v_i \in GF(q)$ are non zero coefficients; $s \leq b - 1, s' \leq b$ and the columns h_i in the second bracket are any b or less consecutive columns among the first $(j - 1 - s)$ columns.

This condition ensures that there shall not be a code vector which can be expressed as sum (difference) of two solid bursts of length b or less each. Thus, the coefficients u_i form a solid burst of length s and the coefficients v_i form a solid burst of length b or less in a $(j - 1 - s)$ -tuple.

The number of choices of these coefficients can be calculated as follows:

If u_i is chosen to be a solid burst of length $(b - 1)$, then the number of solid bursts of length b or less in a $(j - b)$ -tuple, corresponding to the coefficient v_i , is (refer theorem 3)

$$\sum_{i=1}^b (n - b - i + 1)(q - 1)^i$$

If u_i is chosen to be a solid burst of length $(b-2)$, then the number of solid bursts of length b or less in a $(j-b+1)$ -tuple, corresponding to the coefficient v_i , is

$$\sum_{i=1}^b (n-b+1-i+1)(q-1)^i$$

Continuing the process, if u_i is chosen to be a solid burst of length 0, then the number of solid bursts of length b or less in a $(j-1)$ -tuple, corresponding to the coefficient v_i , is

$$(6) \quad \sum_{i=1}^b (j-1-i+1)(q-1)^i$$

Therefore, the total number of possible choices of the coefficients u_i and v_i , are

$$\begin{aligned} & \sum_{i=1}^b (j-1-i+1)(q-1)^i + (q-1) \sum_{i=1}^b (j-2-i+1)(q-1)^i + \\ & \dots + (q-1)^{b-1} \sum_{i=1}^b (j-b-i+1)(q-1)^i \end{aligned}$$

which can be written as

$$(7) \quad \sum_{i=1}^b \sum_{l=1}^b (j-l-i+1)(q-1)^{i+l-1}$$

Thus the column h_j can be added provided

$$(8) \quad q^{n-k} > 1 + \sum_{i=1}^b \sum_{l=1}^b (j-l-i+1)(q-1)^{i+l-1}$$

For a code of length n , replacing j by n gives the theorem. ■

Example 2. Consider a $(9,3)$ binary code with the 6×9 matrix H which has been constructed by the synthesis procedure given in the proof of theorem 4 by taking $b = 3, n = 9$.

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The null space of this matrix can be used to correct all solid bursts of length 3 or less. It may be verified from error pattern-syndromes table 2 that the syndromes of all solid bursts of length 3 or less are non zero and distinct.

Table 2

Error patterns	Syndromes
Solid bursts of length 1	
10000000	10000
01000000	01000
00100000	00100
00010000	00010
00001000	00001
00000100	00000
00000010	10010
00000001	01001
00000001	00100
Solid bursts of length 2	
11000000	11000
01100000	01100
00110000	00110
00011000	00011
00001100	00001
00000110	10010
00000011	11011
00000011	01101
Solid bursts of length 3	
11100000	11100
01110000	01110
00111000	00111
00011100	00011
00001110	10011
00000111	11011
00000011	11111

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