

# HENSTOCK- KURZWEIL INTEGRAL ON $[a,b]$

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## ABSTRACT

The theory of the Riemann integral was not fully satisfactory. Many important functions do not have a Riemann integral. So, Henstock and Kurzweil make the new theory of integral. From the background, the writer will be research about Henstock-Kurzweil integral and also theorems of Henstock- Kurzweil Integral. Henstock- Kurzweil Integral is generalized from Riemann integral. In this case the writer uses research methods literature or literature study carried out by way explore, observe, examine and identify the existing knowledge in the literature. In this thesis explain about partition which used in Henstock-Kurzweil Integral, definition and some property of Henstock- Kurzweil Integral. And some properties of Henstock- Kurzweil integral as follows: value of the Henstock- Kurzweil integral is unique, linearity of the Henstock-Kurzweil integral, Additivity of the Henstock-Kurzweil integral, Cauchy criteria, nonnegativity of Henstock-Kurzweil integral and primitive function.

**Keywords:** *Riemann Integral,  $\delta$  – fine partition, Henstock-Kurzweil Integral.*

## INTRODUCTION

We have already mentioned the developments, during the 1630's, by Fermat and Descartes leading to analytic geometry and the theory of the derivatives. However, the subject we know as calculus did not begin to take shape until the late 1660's when Issac Newton (1642-1727) created his theory of fluxions and invented the method of inverse tangents to find areas under curves. The reversal of the process for finding tangent lines to find areas was also discovered in the 1680's by Leibniz (1646-1716), who was unaware of Newton unpublished work and who arrived at the discovery by a very different route. Leibniz introduced the terminology calculus differential and calculus integral, since finding tangents lines involved differences and finding areas involved summations. Thus they had discovered that integration, being a process of summation, was inverse to the operation of differentiation.

During a century and a half of development and refinement of techniques, calculus consisted of these paired operations and their applications, primarily to physical problems. In the 1850s, Bernhard Riemann (1826-1866) adopted a new and different viewpoint. He separated the concept of integration from its companion, differentiation, and examined the motivating summation and limit process of finding areas by itself. He broadened the scope by considering all functions on an interval for which this process of integration could be defined: the class of integrable functions. The fundamental Theorem of calculus became a result that held only for a restricted set of integrable functions. The

viewpoint of Riemann led others to invent other integration theories, the most significant being Lebesgue's theory of integration.

The theory of the Riemann integral was not fully satisfactory. Many important functions do not have a Riemann integral even after we extend the class of integrable functions slightly by allowing "improper" Riemann integrals. For example Characteristic function.

In 1957, the Czech mathematician Jaroslav Kurzweil discovered a new definition of this integral elegantly similar in nature to Riemann's original definition which he named the gauge integral; the theory was developed by Ralph Henstock. Due to these two important mathematicians, it is now commonly known as Henstock-Kurzweil integral. The simplicity of Kurzweil's definition made some educators advocate that this integral should replace the Riemann integral in introductory calculus courses, but this idea has not gained traction.

Concerning the background of the study, the writer formulates the statement of the problems as follows:

1. How does the concept  $\delta$ -fine partition of Henstock-Kurzweil Integral?
2. How does the definition of Henstock-Kurzweil Integral?
3. How does the fundamental properties of Henstock-Kurzweil Integral?

## REVIEW OF THE RELATED LITERATURE

### 1. Supremum and Infimum

We start with a straightforward definition similar to many others in this course. Read the

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definitions carefully, and note the use of  $\leq$  and  $\geq$  here rather than  $<$  and  $>$ .

**Definition 1**

Let  $S$  be a subset of  $\mathfrak{R}$

1. A number  $u \in \mathfrak{R}$  is said to be an upper bound of  $S$  if  $s \leq u$  for all  $s \in S$
2. A number  $w \in \mathfrak{R}$  is said to be a lower bound of  $S$  if  $w \leq s$  for all  $s \in S$

**Definition 2**

Let  $S$  be a subset of  $\mathfrak{R}$ .

1. If  $S$  is bounded above, then an upper bound  $u$  is said to be *supremum* (or a least upper bound) of  $S$  if no number smaller than  $u$  is an upper bound of  $S$ .
2. If  $S$  is bounded below, then a lower bound  $w$  is said to be *infimum* (or a greatest lower bound) of  $S$  if no number greater than  $w$  is a lower bound of  $S$ .

**2. Limit of Function**

The essence of the concept of limit for real valued functions of a real variable is this: if  $L$  is a real number, then  $\lim_{x \rightarrow x_0} f(x) = L$  means that the value  $f(x)$  can be made as close to  $L$  as we wish by taking  $x$  sufficiently close to  $x_0$ . This is made precise in the following definition.

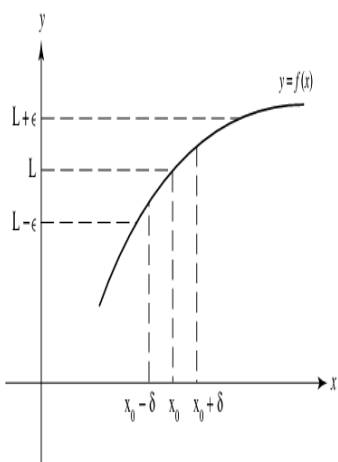


Figure 1. The limit of  $f$  at  $x_0$  is  $L$

**Definition 3**

We say that  $f(x)$  approaches the limit  $L$  as  $x$  approaches  $x_0$ , and write

$$\lim_{x \rightarrow x_0} f(x) = L$$

If  $f$  is defined on some deleted neighborhood of  $x_0$ , and for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that

$$|f(x) - L| < \epsilon$$

if

$$0 < |x - x_0| < \delta$$

**3. Compact Sets**

**Definition 4**

A subset  $K$  of  $R$  is said to be compact if every open cover of  $K$  has a finite sub cover.

In other words, a set  $K$  is compact if, whenever it is contained in the union of a collection  $G = \{G_\alpha\}$  of open sets in  $R$ , then it is contained in the union of some finite number of sets in  $G$ .

**Theorem 1**

If  $K$  is a compact subset of  $R$ , then  $K$  is closed and bounded.

**Proof:**

We shall first show that  $K$  is bounded. For each  $m \in N$ , let  $H_m := (-m, m)$ . Since each  $H_m$  is open and since  $K \subseteq \cup_{m=1}^M H_m = R$ , we see that the collection  $\{H_m : m \in N\}$  is an open cover of  $K$ . since  $K$  is compact, this collection has a finite sub cover, so there exists  $M \in N$  such that,

$$K \subseteq \bigcup_{m=1}^M H_m = H_M = (-M, M)$$

Therefore  $K$  is bounded, since it is contained in the bounded interval  $(-M, M)$ .

We show that  $K$  is closed, by showing that its complement  $u = \mathcal{C}(K)$  is open. To do so, let  $u = \mathcal{C}(K)$  be arbitrary and for each  $n \in N$ , we let  $G_n := \{y \in R : |y - u| > 1/n\}$ . Since  $u \notin K$ , we have  $K \subseteq \cup_{n=1}^\infty G_n$ . Since  $K$  is compact, there exists  $m \in N$  such that

$$K \subseteq \bigcup_{n=1}^m G_n = G_m$$

Now it follows from this that

$$K \cap (u - 1/m, u + 1/m) = \emptyset,$$

so that  $(u - 1/m, u + 1/m) \subseteq \mathcal{C}(K)$ . but since  $u$  was an arbitrary point in  $\mathcal{C}(K)$ , we infer that  $\mathcal{C}(K)$  is open.

**4. Continuity**

**Definition 5**

- a) We say that  $f$  is continuous at  $x_0$  if  $f$  is defined on an open interval  $(a,b)$  containing  $x_0$  and  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ .
- b) We say that  $f$  is continuous from the left at  $x_0$  if  $f$  is defined on an open interval  $(a, x_0)$  and  $f(x_0 -) = f(x_0)$ .
- c) We say that  $f$  is continuous from the right at  $x_0$  if  $f$  is defined on an open interval  $(x_0, b)$  and  $f(x_0 +) = f(x_0)$ .

**Theorem 2**

- a) A function  $f$  is continuous at  $x_0$  if and only if  $f$  is defined on an open interval  $(a, b)$  containing  $x_0$  and for each  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $|f(x) - f(x_0)| < \varepsilon$ , whenever  $|x - x_0| < \delta$
- b) A function  $f$  is continuous from the right at  $x_0$  if and only if  $f$  is defined on an open interval  $[x_0, b)$  and for each  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $|f(x) - f(x_0)| < \varepsilon$  holds whenever  $x_0 \leq x < x_0 + \delta$
- c) A function  $f$  is continuous from the left at  $x_0$  if and only if  $f$  is defined on an open interval  $(a, x_0]$  and for each  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $|f(x) - f(x_0)| < \varepsilon$  holds whenever  $x_0 - \delta \leq x < x_0$

**Definition 6**

A function  $f$  is continuous on an open interval  $(a, b)$  if it is continuous at every point in  $(a, b)$ . if, in addition.

$$f(b-) = f(b) \text{ or } f(a+) = f(a)$$

Then  $f$  is continuous on  $(a, b)$  or  $[a, b)$ , respectively, if  $f$  continuous on  $(a, b)$  and  $f(b-) = f(b)$  or  $f(a+) = f(a)$  both hold, then  $f$  is continuous on  $[a, b]$ .

**Definition 7**

A function  $f$  is piecewise continuous on  $[a, b]$  if

- a)  $f(x_0+)$  exist for all  $x_0$  in  $[a, b)$ ;
- b)  $f(x_0-)$  exist for all  $x_0$  in  $(a, b]$ ;
- c)  $f(x_0+) = f(x_0-) = f(x_0)$  for all but finitely many points  $x_0$  in  $(a, b)$ .

If c) fails to hold at some  $x_0$  in  $(a, b)$ ,  $f$  has a jump discontinuity at  $x_0$ . Also,  $f$  has a jump discontinuity at  $a$  if  $f(a+) \neq f(a)$  or at  $b$  if  $f(b-) \neq f(b)$ .

**5. Uniform Continuity****Definition 8**

Let  $A \subseteq \mathbb{R}$ , let  $f : A \rightarrow \mathbb{R}$ , we say that  $f$  is uniformly continuous on  $A$  if for each  $\varepsilon > 0$  there

is a  $\delta(\varepsilon) > 0$  such that if  $x, u \in A$  are any number satisfying, then  $|f(x) - f(u)| < \varepsilon$ .

**Theorem 3**

If  $f$  is continuous on a closed interval  $[a, b]$ , then  $f$  is uniformly continuous on  $[a, b]$ .

**Proof:**

Suppose that  $\varepsilon > 0$ . Since  $f$  is continuous on  $[a, b]$ , for each  $t$  in  $[a, b]$  there is a positive number  $\delta_t$  such that

$$|f(x) - f(t)| < \frac{\varepsilon}{2}$$

$$\text{if } |x - t| < 2\delta_t \\ \text{and } x \in [a, b]$$

If

$$I_t = (t - \delta_t, t + \delta_t),$$

the collection

$$H = \{I_t | t \in [a, b]\}$$

is an open covering of  $[a, b]$ . Since  $[a, b]$  is compact, the Heine-Borel theorem implies that there are finitely many points  $t_1, t_2, \dots, t_n$  in  $[a, b]$  such that  $I_{t_1}, I_{t_2}, \dots, I_{t_n}$  cover  $[a, b]$ .

Now define

$$\delta = \min\{\delta_{t_1}, \delta_{t_2}, \dots, \delta_{t_n}\}$$

We will show that if

$$|x - x'| < \delta \\ \text{and } x, x' \in [a, b]$$

Then

$$|f(x) - f(x')| < \varepsilon$$

From the triangle inequality.

$$|f(x) - f(x')| = |(f(x) - f(t_r)) + (f(t_r) - f(x'))|$$

$$\leq |(f(x) - f(t_r))| + |(f(t_r) - f(x'))|$$

Since  $I_{t_1}, I_{t_2}, \dots, I_{t_n}$  cover  $[a, b]$ ,  $x$  must be in one of these intervals. Suppose that  $x \in I_{t_r}$ ;

That is,

$$|x - t_r| < \delta_{t_r}$$

With

$$t = t_r,$$

$$|f(x) - f(t_r)| < \frac{\varepsilon}{2}.$$

Such that,

$$|x'-t_r| = |(x'-x) + (x-t_r)| \leq |x'-x| + |x-t_r| < \gamma + \delta_r \leq 2\delta_r. \quad s(P) = \sum_{j=1}^n m_j (x_j - x_{j-1})$$

Therefore with  $t = t_r$ , and  $x$  replaced by  $x'$  implies that

$$|f(x') - f(t_r)| < \frac{\epsilon}{2}.$$

This imply that  $|f(x) - f(x')| < \epsilon$ .

**Definition 9 (Lipschitz Functions)**

Let  $A \subseteq \mathbb{R}$ , let  $f : A \rightarrow \mathbb{R}$ . If there exist a constant  $K > 0$  such that

$$|f(x) - f(u)| \leq K|x - u|$$

For all  $x, u \in A$ , then  $f$  is said to be a **Lipschitz Functions** on  $A$

**Theorem 4**

If  $f : A \rightarrow \mathbb{R}$  is a Lipschitz Functions, then  $f$  is uniform continuous on  $A$ .

**Proof:**

If the a Lipschitz conditions satisfied with constant  $K$ , then given  $\epsilon > 0$ , we can take

$$\delta := \frac{\epsilon}{K}. \text{ If } x, u \in A \text{ satisfy } |x - u| < \delta, \text{ then}$$

$$|f(x) - f(u)| < K \cdot \frac{\epsilon}{K} = \epsilon$$

Therefore  $f$  is uniformly continuous on  $A$ .

**6. Upper And Lower Integral**

**Definition 10**

If  $f$  is bounded on  $[a, b]$  and  $P\{x_0, x_1, \dots, x_n\}$  is a partition of  $[a, b]$ , let

$$M_j = \sup_{x_{j-1} \leq x \leq x_j} f(x)$$

And

$$m_j = \inf_{x_{j-1} \leq x \leq x_j} f(x)$$

The upper sum of  $f$  over  $P$  is

$$S(P) = \sum_{j=1}^n M_j (x_j - x_{j-1})$$

And the upper integral of  $f$  over  $[a, b]$ , denoted by

$$\int_a^b f(x) dx$$

Is the infimum of all upper sums. The lower sum of  $f$  over  $P$  is

And the lower integral of  $f$  over  $[a, b]$ , denoted by

$$\int_a^b f(x) dx$$

Is the supremum off all lower sums.

**7. Riemann Integral**

Riemann integral, defined in 1854, was the first of the modern theories of integration and enjoys many of the desirable properties of an integration theory. The groundwork for the Riemann integral of a function  $f$  over the interval  $[a, b]$  begins with dividing the interval into smaller subintervals.

With infimum and supremum taken include all partitions  $P$  on  $[a, b]$ , if the upper integral and lower integral same, then  $f$  can be said integrable on  $[a, b]$ . And called Riemann function  $f$  on  $[a, b]$  and denoted by  $f \in [a, b]$ . This same value is called the Riemann integral function  $f$  on  $f \in [a, b]$  and written

$$(R) \int_a^b f(x) dx$$

**Definition 11**

Let  $[a, b] \subset \mathbb{R}$ . A partition of  $[a, b]$  is a finite set of numbers  $P = \{x_0, x_1, \dots, x_n\}$  such that  $x_0 = a, x_n = b$  and  $x_{i-1} < x_i$  for  $i = 1, 2, \dots, n$ . For each subinterval  $[x_{i-1}, x_i]$ , define its length to be  $\ell([x_{i-1}, x_i]) = x_i - x_{i-1}$ . The mesh of the partition is then the length of the largest subinterval,  $[x_{-1}, x_i]$ :

$$\mu(P) = \max\{x_i - x_{i-1} : i = 1, 2, \dots, n\}$$

Thus the point  $\{x_0, x_1, \dots, x_n\}$  form an increasing sequence of numbers in  $[a, b]$  that divides the interval  $[a, b]$  into contiguous subintervals.

Let  $f : [a, b] \rightarrow \mathbb{R}$ ,  $P = \{x_0, x_1, \dots, x_n\}$  be a partition of  $[a, b]$ , and  $t_i \in [x_{i-1}, x_i]$  for each  $i$ . Riemann began by considering the approximating (Riemann) sums

$$S(f, P, \{t_i\}_{i=1}^n) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}),$$

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Defined with respect to the partition  $P$  and the set of sampling points  $\{t_i\}_{i=1}^n$ . Riemann considered the integral of  $f$  over  $[a, b]$  to be a "limit" of the sums  $S(f, P, \{t_i\}_{i=1}^n)$ , in the following sense.

**Definition 12**

A function  $f : [a, b] \rightarrow \mathfrak{R}$  is Riemann integrable over  $[a, b]$  if there is an  $A \in \mathfrak{R}$  such that for all  $\varepsilon > 0$  there is a  $\delta > 0$  so that if  $P$  is any partition of  $[a, b]$  with  $\mu(P) < \delta$  and  $t_i \in [x_{i-1}, x_i]$  for all  $i$  then

$$|S(f, P, \{t_i\}_{i=1}^n) - A| < \varepsilon$$

We write  $A = \int_a^b f = \int_a^b f(t)dt$  or, if we set

$$I = [a, b], \int_I f.$$

This definition defines the integral as a limit of sums as the mesh of the partition approaches 0.

**DISCUSSION**

**1. Concept  $\delta$ -fine Partition of Henstock-Kurzweil Integral**

Let  $[a, b]$  be a compact interval in  $\mathfrak{R}$ . Let  $D$  be a finite collection of interval-point pairs  $\{([u_i, v_i], \xi_i)\}_{i=1}^n$ , where  $\{([u_i, v_i])\}_{i=1}^n$  are non-overlapping subintervals of  $[a, b]$ . Let  $\delta(\xi)$  be a positive function on  $[a, b]$ , i.e.  $\delta(\xi) : [a, b] \rightarrow \mathfrak{R}^+$ . We say

$D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$  is  $\delta$ -fine Henstock-Kurzweil partition of  $[a, b]$  if  $\xi_i \in [u_i, v_i] \subset B(\xi_i, \delta(\xi_i)) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$  for all  $i = 1, 2, 3, \dots, n$

Given an  $\delta$ -fine Henstock-Kurzweil partition  $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$  we write

$$S(f, D) = \sum_{i=1}^n f(\xi_i)(v_i - u_i)$$

For integral sum over  $D$ , whenever  $f : [a, b] \rightarrow \mathfrak{R}$

**Example 1**

Let  $f(x) = x$  Consider a division  $a = x_0 < x_1 < \dots < x_n = b$  and  $\{\xi_1, \xi_2, \dots, \xi_n\}$ .

And this time choose the points  $\xi_i = \frac{1}{2}(x_i + x_{i-1})$ , Clearly  $\xi_i \in [x_{i-1}, x_i]$  For  $i = 1, 2, \dots, n$

Then,

$$\begin{aligned} S(f, D) &= \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^n \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \frac{1}{2} \sum_{i=1}^n (x_i^2 - x_{i-1}^2) \\ &= \frac{1}{2}(x_n^2 - x_0^2) \\ &= \frac{1}{2}(b^2 - a^2) \end{aligned}$$

Then,

$$S(f, D) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) = \frac{1}{2}(b^2 - a^2)$$

**2. Definition Of Henstock-Kurzweil Integral**

**Definition 13**

A function  $f : [a, b] \rightarrow \mathfrak{R}$  is said to be Henstock-Kurzweil integrable on  $[a, b]$  if there exists a real number  $A$  such that for every  $\varepsilon > 0$  there exists  $\delta : [a, b] \rightarrow \mathfrak{R}^+$  such that for every  $\delta$ -fine Henstock-Kurzweil partition  $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$  of  $[a, b]$ , we have

$$\left| \sum_{i=1}^n f(\xi_i)(v_i - u_i) - A \right| < \varepsilon$$

We denote the Henstock-Kurzweil Integral (also write as HK-integral)  $A$  by  $(HK) \int_a^b f(x)dx$ .

**Example 2**

Define  $f : [0, 1] \rightarrow \mathfrak{R}$  the Dirichlet's function (= the characteristic function of the rational numbers in  $[0, 1]$ ), by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathcal{Q} \\ 0 & \text{if } x \notin \mathcal{Q} \end{cases}$$

Then  $f(x)$  is Henstock-Kurzweil integrable on  $[0, 1]$ . And  $\int_0^1 f(x) = 0$

To Prove this assertion, we will define an appropriate gauge  $\delta_\varepsilon$ , First we enumerate these rational numbers as  $r_1, r_2, \dots$ . We define

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$\delta(r_i) = \varepsilon 2^{-i-1}$  for  $i = 1, 2, \dots$ , and if  $x \in [0, 1]$  is irrational we define  $\delta(\xi) = 1$ ; clearly  $\delta_\varepsilon$  is a gauge on  $[0, 1]$ . If  $P$  is a  $\delta$ -fine tagged partition, there can be at most two subintervals in  $P$  that have the number  $r_i$  as tag, and the length of each of those subintervals is  $\leq \varepsilon 2^{-i-1}$ . Hence the contribution to  $S(f, P)$  from subintervals with tag  $r_i$  is  $\leq \varepsilon 2^{-i}$ . Since the terms in  $S(f, P)$  with tags at irrational points contribute 0, we readily see that

$$0 \leq S(f, P) < \sum_{i=1}^{\infty} \varepsilon / 2^i = \varepsilon$$

Since  $\varepsilon > 0$  is arbitrary, this shows that  $f(x)$  is Henstock-Kurzweil integrable on  $[0, 1]$ . And

$$\int_0^1 f(x) = 0$$

### 3. Fundamental Properties Of Henstock-Kurzweil Integral

#### Theorem 5 (Unique Property)

if  $f$  is Henstock-Kurzweil integrable over  $[a, b]$ , then the value of the integral is unique.

**Proof:**

Suppose that  $f$  is Henstock-Kurzweil integrable on  $[a, b]$  and both real number  $A$  and  $B$  satisfy Definition 3.2.1. Fix  $\varepsilon > 0$  choose  $\delta_A$  and  $\delta_B$  corresponding to  $A$  and  $B$ , respectively, in the definition with  $\varepsilon' = \frac{\varepsilon}{2}$ . Let  $\delta = \min(\delta_A, \delta_B)$

and suppose for every  $\delta$ -fine Henstock-Kurzweil partition  $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$  of  $[a, b]$ , Then

$$\begin{aligned} |A - B| &= \left| \left( \sum_{i=1}^n f(\xi_i)(v_i - u_i) - B \right) - \left( \sum_{i=1}^n f(\xi_i)(v_i - u_i) - A \right) \right| \\ &\leq \left| A - \sum_{i=1}^n f(\xi_i)(v_i - u_i) \right| + \left| \sum_{i=1}^n f(\xi_i)(v_i - u_i) - B \right| \\ &< \varepsilon' + \varepsilon' = \varepsilon \end{aligned}$$

Since  $\varepsilon$  was arbitrary, it follows that  $A = B$ . Thus, the value of the integral is unique.

#### Theorem 6 (Linearity of the Henstock-Kurzweil integral)

If  $f$  and  $g$  are Henstock-Kurzweil integrable on  $[a, b]$ , then so are  $f + g$  and  $\alpha f$  where  $\alpha$  is real. Furthermore,

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g \quad \text{and} \quad \int_a^b (\alpha f) = \alpha \int_a^b f$$

**Proof:** Let  $A$  and  $B$  denote respectively the integrals of  $f$  and  $g$  on  $[a, b]$ . given  $\varepsilon > 0$ , there is a  $\delta_1(\xi) > 0$  such that for any  $\delta_1$ -fine division  $D = ([u, v], \xi)$  we have

$$\left| \sum f(\xi)(v - u) - A \right| < \frac{\varepsilon}{2}$$

Similarly, there is a  $\delta_2(\xi) > 0$  such that for any  $\delta_2$ -fine division  $D = ([u, v], \xi)$  we have

$$\left| \sum g(\xi)(v - u) - B \right| < \frac{\varepsilon}{2}$$

Now put  $\delta(\xi) = \min(\delta_1(\xi), \delta_2(\xi))$ . Note that any  $\delta$ -fine division is also  $\delta_1$ -fine and  $\delta_2$ -fine. Therefore for any  $\delta$ -fine division  $D = ([u, v], \xi)$  we have

$$\left| \sum (f(\xi) + g(\xi))(v - u) - (A + B) \right| \leq$$

$$\left| \sum f(\xi)(v - u) - A \right| + \left| \sum g(\xi)(v - u) - B \right| < \varepsilon$$

The proof is complete.

#### Example 3

evaluate  $\int_0^1 f + g$  where  $f(x) = x^2$  and

$$g(x) = x.$$

Solution:

By theorem 3.3.3,

$$\int_0^1 f + g = \int_0^1 x^2 + x dx = \int_0^1 x^2 dx + \int_0^1 x dx$$

Now, we evaluate value  $\int_0^1 x^2 dx$

Consider a division  $0 = x_0 < x_1 < \dots < x_n = 1$  and  $\{\xi_1, \xi_2, \dots, \xi_n\}$ . And this time choose the intermediate points

$$\xi_i = \left[ \frac{1}{3} (x_i^2 + x_i x_{i-1} + x_{i-1}^2) \right]^{1/2},$$

then

$$0 \leq x_{i-1} = (x_{i-1}^2)^{1/2} < \left[ \frac{1}{3} (x_i^2 + x_i x_{i-1} + x_{i-1}^2) \right]^{1/2} < (x_i^2)^{1/2} = x_i$$

For  $i = 1, 2, \dots, n$ ; that is  $\xi_i \in (x_{i-1}, x_i)$  for each  $i$ .

So

$$\int_0^1 x^2 dx = \frac{1}{3}$$

And then, we evaluate value  $\int_0^1 x dx$

Consider a division  $0 = x_0 < x_1 < \dots < x_n = 1$  and  $\{\xi_1, \xi_2, \dots, \xi_n\}$ . And this time choose the points

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$\xi_i = \frac{1}{2}(x_i + x_{i-1})$ , Clearly  $\xi_i \in [x_{i-1}, x_i]$  For  $i = 1, 2, \dots, n$

Now

$$\begin{aligned} S(f, D) &= \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^n \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \frac{1}{2} \sum_{i=1}^n (x_i^2 - x_{i-1}^2) \\ &= \frac{1}{2}(x_n^2 - x_0^2) \\ &= \frac{1}{2} \cdot 1^2 \\ &= \frac{1}{2} \cdot 1 = \frac{1}{2} \end{aligned}$$

So,  $\int_0^1 x dx = \frac{1}{2}$  that,

$$\int_0^1 f + g = \int_0^1 x^2 + x dx = \int_0^1 x^2 dx + \int_0^1 x dx = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

**Theorem 7 (Additivity of the henstock-Kurzweil Integral)**

Let  $a < c < b$ . If  $f$  is Henstock-Kurzweil integrable on  $[a, c]$  and on  $[c, b]$  and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

**Proof:**

Let A denote the integral of  $f$  on  $[a, c]$  and B that of  $f$  on  $[c, b]$ . Given  $\epsilon > 0$ , there is a  $\delta_1(\xi) > 0$ , defined on  $[a, c]$ , such that for any  $\delta_1$ -fine division  $D = ([u, v]; \xi)$  of  $[a, c]$  we have

$$|\sum f(\xi)(v-u) - A| < \frac{\epsilon}{2}$$

Similarly, there is a  $\delta_2(\xi) > 0$  defined on  $[c, b]$  such that for any  $\delta_2$ -fine division  $D = ([u, v]; \xi)$  of  $[c, b]$  we have

$$|\sum f(\xi)(v-u) - B| < \frac{\epsilon}{2}$$

Define  $\delta(\xi) = \min(\delta_1(\xi), c - \xi)$  when  $\xi \in [a, c)$ ,  $\min(\delta_2(\xi), \xi - c)$  when  $\xi \in (c, b]$ , and  $\min(\delta_1(c), \delta_2(c))$  when  $\xi = c$ . Note for any  $\delta$ -fine division D of  $[a, b]$ ,  $c$  is always a division point of D. therefore for any  $\delta$ -fine division  $D = ([u, v]; \xi)$  of  $[a, b]$  with  $\Sigma$  over D, writing  $\Sigma = \Sigma_1 + \Sigma_2$

where  $\Sigma_1$  is the partial sum over  $[a, c]$  and  $\Sigma_2$  over  $[c, b]$  we have

$$|\sum f(\xi)(v-u) - (A+B)| \leq$$

$$|\sum_1 f(\xi)(v-u) - A| + |\sum_2 f(\xi)(v-u) - B| < \epsilon$$

Hence  $f$  is Henstock-Kurzweil integrable to  $A+B$  on  $[a, b]$ .

Alternatively, Let  $\chi_{[a,c]}$  denote the characteristic function of  $[a, c]$  and  $f_1 = f\chi_{[a,c]}$ . Similarly, let  $f_2 = f\chi_{[c,b]}$ . Then it follows from Theorem 3.3.1 that

$$\int_a^b f = \int_a^b (f_1 + f_2) = \int_a^c f + \int_c^b f$$

**Example 7**

Let  $f(x) = x$  and Let  $0 < \frac{1}{2} < 1$ . and, If  $f$  is

Henstock-Kurzweil integrable on  $[a, c] = [0, \frac{1}{2}]$

and on  $[c, b] = [\frac{1}{2}, 1]$  and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Solution:

Consider a division  $0 = x_0 < x_1 < \dots < x_n = 1$  and  $\{\xi_1, \xi_2, \dots, \xi_n\}$ . And this time choose the points

$\xi_i = \frac{1}{2}(x_i + x_{i-1})$ , Clearly  $\xi_i \in [x_{i-1}, x_i]$  For  $i = 1, 2, \dots, n$

Now

$$\begin{aligned} S(f, D) &= \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^n \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \frac{1}{2} \sum_{i=1}^n (x_i^2 - x_{i-1}^2) \\ &= \frac{1}{2}(x_n^2 - x_0^2) \\ &= \frac{1}{2}(1^2 - 0^2) \\ &= \frac{1}{2}(1 - 0) \end{aligned}$$

With same procedure we get; on

$$[a, c] = [0, \frac{1}{2}]$$

$$\begin{aligned} S(f, D) &= \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^n \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \frac{1}{2} \sum_{i=1}^n (x_i^2 - x_{i-1}^2) \\ &= \frac{1}{2}(x_n^2 - x_0^2) \\ &= \frac{1}{2}(c^2 - a^2) \end{aligned}$$

So  $\varepsilon > 0$ , there is a  $\delta_1(\xi) > 0$ , defined on  $[0, \frac{1}{2}]$ , such that for any  $\delta_1$ -fine division  $D = ([u, v]; \xi)$  of  $[0, \frac{1}{2}]$  we have

$$\left| \frac{1}{2}(c^2 - a^2) - \frac{1}{8} \right| < \frac{\varepsilon}{2}$$

And on  $[c, b] = [\frac{1}{2}, 1]$ , we get

$$\begin{aligned} S(f, D) &= \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^n \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \frac{1}{2} \sum_{i=1}^n (x_i^2 - x_{i-1}^2) \\ &= \frac{1}{2}(x_n^2 - x_0^2) \\ &= \frac{1}{2}(b^2 - c^2) \end{aligned}$$

$\varepsilon > 0$ , there is a  $\delta_2(\xi) > 0$ , defined on  $[\frac{1}{2}, 1]$ , such that for any  $\delta_2$ -fine division  $D = ([u, v]; \xi)$  of  $[\frac{1}{2}, 1]$  we have

$$\left| \frac{1}{2}(b^2 - c^2) - \frac{3}{8} \right| < \frac{\varepsilon}{2}$$

therefore for any  $\delta$ -fine division  $D = ([u, v]; \xi)$  of  $[0, 1]$

with  $\Sigma$  over  $D$ , writing  $\Sigma = \Sigma_1 + \Sigma_2$  where  $\Sigma_1$  is the partial sum over  $[a, c]$  and  $\Sigma_2$  over  $[\frac{1}{2}, 1]$  we have

$$\begin{aligned} \left| \frac{1}{2}(b^2 - a^2) - \left( \frac{1}{8} + \frac{3}{8} \right) \right| &\leq \\ \left| \sum_1 f(\xi)(v-u) - \frac{1}{8} \right| + \left| \sum_2 f(\xi)(v-u) - \frac{3}{8} \right| &< \varepsilon \end{aligned}$$

Hence  $f$  is Henstock-Kurzweil integrable to  $\frac{1}{2}$  on  $[0, 1]$

**Lemma 8 (Cauchy Criteria)**

A function is Henstock-kurzweil integrable on  $[a, b]$  if and only if for every  $\varepsilon > 0$ , there is a  $\delta(\xi) > 0$  such that for any  $\delta$ -fine division  $D = ([u, v]; \xi)$  and  $D' = ([u', v']; \xi')$  we have

$$\left| \sum f(\xi)(v-u) - \sum f(\xi')(v'-u') \right| < \varepsilon$$

Where the first sum is over  $D$  and the second over  $D'$ .

**Proof**

( $\Rightarrow$ ) we will prove that if A function is Henstock-kurzweil integrable on  $[a, b]$  Then for every  $\varepsilon > 0$ , there is a  $\delta(\xi) > 0$  such that for any  $\delta$ -fine division  $D = ([u, v]; \xi)$  and  $D' = ([u', v']; \xi')$  we have

$$\left| \sum f(\xi)(v-u) - \sum f(\xi')(v'-u') \right| < \varepsilon$$

A function is Henstock-kurzweil integrable on  $[a, b]$  Then for every  $\varepsilon > 0$ , there is a  $\delta(\xi) > 0$  such that for any  $\delta$ -fine division  $D = ([u, v]; \xi)$  and  $D' = ([u', v']; \xi')$

$$\begin{aligned} \left| \sum f(\xi)(v-u) - \sum f(\xi')(v'-u') \right| &= \left| \sum f(\xi)(v-u) - A + A - \sum f(\xi')(v'-u') \right| \\ &\leq \left| \sum f(\xi)(v-u) - A \right| + \left| A - \sum f(\xi')(v'-u') \right| \\ &< \varepsilon \end{aligned}$$

Analogous to the situation for real-valued sequences, the condition that

$$\left| \sum f(\xi)(v-u) - \sum f(\xi')(v'-u') \right| < \varepsilon$$

( $\Leftarrow$ ) We have already proved that the integrability of  $f$  implies the Cauchy criterion. So, assume the Cauchy criterion holds. We will prove that  $f$  is Henstock-kurzweil integrable.

if for every  $\varepsilon > 0$ , there is a  $\delta(\xi) > 0$  such that for any  $\delta$ -fine division  $D = ([u, v]; \xi)$  and  $D' = ([u', v']; \xi')$  we have

$$\left| \sum f(\xi)(v-u) - \sum f(\xi')(v'-u') \right| < \varepsilon$$

Then A function is Henstock-Kurzweil integrable on  $[a, b]$ .

For each  $k \in \mathbb{N}$ , choose a  $\delta_k > 0$  so that for any two division  $D = ([u, v]; \xi)$  and  $D' = ([u', v']; \xi')$ , and corresponding sampling points, we have

$$\left| \sum f(\xi)(v-u) - \sum f(\xi')(v'-u') \right| < \frac{1}{k}$$

Replacing  $\delta_k$  by  $\min\{\delta_1, \delta_2, \dots, \delta_k\}$ , we may assume that  $\delta_k \geq \delta_{k+1}$ .

Next for each  $k$ , fix a partition  $D_k = ([u_k, v_k]; \xi_k)$  and set of sampling point  $\{\xi_i\}_{i=1}^n$ . Note for  $j > k$  Thus

$$\left| \sum f(\xi_k)(v_k - u_k) - \sum f(\xi_j)(v_j - u_j) \right| < \frac{1}{\min\{j, k\}}$$



Which implies that sequence  $\sum_{k=1}^{\infty} f(\xi_k)(v_k - u_k)$

is a Cauchy sequence in  $R$ , and hence converges. Let  $A$  be a limit of this sequence. it follows from the previous inequality that

$$\left| \sum f(\xi)(v-u) - A \right| < \frac{1}{k}$$

It remains to show that  $A$  satisfies Definition 3.2.1  
Fix  $\varepsilon > 0$  and let division  $D = ([u, v]; \xi)$ . Then

$$\begin{aligned} & \left| \sum f(\xi)(v-u) - A \right| \\ &= \left| \sum_{i=1}^n f(\xi_i)(v_i - u_i) - \sum_{k=1}^n f(\xi_k)(v_k - u_k) + \sum_{k=1}^n f(\xi_k)(v_k - u_k) - A \right| \\ &\leq \left| \sum_{i=1}^n f(\xi_i)(v_i - u_i) - \sum_{k=1}^n f(\xi_k)(v_k - u_k) \right| + \left| \sum_{k=1}^n f(\xi_k)(v_k - u_k) - A \right| \\ &< \frac{1}{k} + \frac{1}{k} = \varepsilon \end{aligned}$$

It now follows that  $f$  is Henstock-Kurzweil integrable on  $[a, b]$

**Theorem 9**

If  $f$  is Henstock-Kurzweil integrable on  $[a, b]$ , then so it is on a subinterval  $[c, d]$  of  $[a, b]$ .

**Proof:**

Since  $f$  is Henstock-Kurzweil integrable on  $[a, b]$ , the Cauchy condition holds. Take any two  $\delta$ -fine divisions of  $[c, d]$ . say  $D_1$  and  $D_2$ , and denote by  $s_1$  and  $s_2$  respectively the Riemann sums of  $f$  over  $D_1$  and  $D_2$ . similarly, take another  $\delta$ -fine division  $D_3$  of  $[a, c] \cup [d, b]$  and denote by  $s_3$  the corresponding Riemann sums. Then the union  $D_1 \cup D_3$  forms a  $\delta$ -fine division of  $[a, b]$ . Here the division points and associated points of  $D_1 \cup D_3$  are the union of those from  $D_1$  and  $D_3$ .

The Riemann sum of  $f$  over  $D_1 \cup D_3$  is  $s_1 + s_3$ .

And similarly that over  $D_2 \cup D_3$  is  $s_2 + s_3$

.Therefore by the Cauchy condition we have

$$|s_1 - s_2| \leq |(s_1 + s_3) - (s_2 + s_3)| < \varepsilon$$

Hence the result follows from lemma 3.3.7 with  $[a, b]$  replaced by  $[c, d]$

**Theorem 10 (Nonnegativity of The Henstock-Kurzweil integral)**

If  $f$  and  $g$  are Henstock-Kurzweil integrable on  $[a, b]$  and if  $f(x) \leq g(x)$  for almost all in  $x$  in  $[a, b]$ , then

$$\int_a^b f \leq \int_a^b g$$

**Proof:**

In view of theorem 3.3.9 we may assume that  $f(x) \leq g(x)$  for all  $x$ . Given  $\varepsilon > 0$ , as in the proof of theorem 3.2.1, there is a  $\delta(\xi) > 0$  such that for any  $\delta$ -fine division  $D = ([u, v]; \xi)$  we have

$$\begin{aligned} & \left| \sum f(\xi)(v-u) - \int_a^b f \right| < \varepsilon \\ & \left| \sum g(\xi)(v-u) - \int_a^b g \right| < \varepsilon \end{aligned}$$

It follows that

$$\int_a^b f - \varepsilon < \sum f(\xi)(v-u) < \sum g(\xi)(v-u) < \int_a^b g + \varepsilon$$

Since  $\varepsilon$  is arbitrary, we have the required inequality.

**Theorem 11**

If  $f$  is Henstock-Kurzweil integrable on  $[a, b]$  with the primitive  $F$ , then for every  $\varepsilon > 0$ , there is a  $\delta(\xi) > 0$  such that for any  $\delta$ -fine division  $D = ([u, v]; \xi)$  we have

$$\sum |F(v) - F(u) - f(\xi)(v-u)| < \varepsilon$$

We shall make a few remarks. Before proof, from the computational point of view, we may regard  $f(\xi)(v-u)$  as an approximation of  $F(v) - F(u)$ . Then the difference  $F(v) - F(u) - f(\xi)(v-u)$  is an error. The definition of the Henstock-Kurzweil integral says that the absolute error is also small, whereas Henstock'Lemma. In fact, the two are equivalent by theorem 3.3.8. Another way of putting it is that taking any partial sum  $\Sigma_1$  of  $\Sigma$  we still have

$$|\Sigma_1 F(v) - F(u) - f(\xi)(v-u)| < \varepsilon$$

That is to say, the selected error is again small, and indeed it is equivalent to the above two.

**Proof:**

Given  $\varepsilon > 0$ , there is a  $\delta(\xi) > 0$  such that for any  $\delta$ -fine division  $D = ([u, v]; \xi)$  we have

$$|\Sigma F(v) - F(u) - f(\xi)(v-u)| < \varepsilon/4$$

Let  $\Sigma_1$  be a partial sum of  $\Sigma$  and  $E_1$  the union of  $[u, v]$  from  $\Sigma_1$ . Suppose  $E_2$ . thus we can choose a  $\delta$ -fine division  $D_2 = ([u, v]; \xi)$  of  $E_2$  such that

$$|\Sigma_2 F(v) - F(u) - f(\xi)(v-u)| < \varepsilon/4$$

Where  $\Sigma_2$  is over  $D_2$ . Now writing  $\Sigma_3 = \Sigma_1 + \Sigma_2$  we have

$$|\Sigma(F(v) - F(u) - f(\xi)(v-u))| \leq$$

$$|\Sigma_1(F(v) - F(u) - f(\xi)(v-u))| + |\Sigma_2(F(v) - F(u) - f(\xi)(v-u))| < \varepsilon/2$$

Consequently the result follows.

**Example 8**

Let  $f(x) = 1/\sqrt{x}$  for  $0 < x \leq 1$ . Given  $\varepsilon > 0$ ,

we shall construct  $\delta(\xi)$  so that  $f$  is Henstock-Kurzweil integrable on  $[0,1]$ . Consider a division

$$0 = x_0 < x_1 < \dots < x_n = 1 \text{ and } \{\xi_1, \xi_2, \dots, \xi_n\}$$

With  $\xi_1 = 0$  and  $x_{i-1} \leq \xi_i \leq x_i$  for  $i = 2, \dots, n$ .

Note that the primitive of  $1/\sqrt{x}$  is  $2\sqrt{x}$ . Then

we can write

$$\begin{aligned} & \left| 2 - \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \right| \leq \\ & \left| 2 - (2 - \sqrt{x_1}) \right| + \left| \int_{x_1}^1 dx/\sqrt{x} - \sum_{i=2}^n (x_i - x_{i-1})/\sqrt{\xi_i} \right| \leq \\ & 2\sqrt{x_1} + \sum_{i=2}^n (1/\sqrt{x_{i-1}} - 1/\sqrt{x_i})(x_i - x_{i-1}). \end{aligned}$$

We shall prove that above is less than  $\varepsilon$  for suitable  $\delta$  - fine divisions. Suppose  $\delta(\xi) = c\xi$  for  $0 < \xi \leq 1$  and  $0 < c < 1/2$  so that  $\xi_1 = 0$  always. If the above division is  $\delta$  - fine and  $[u, v]$  is a typical interval  $[x_{i-1}, x_i]$  in the division with  $u \neq 0$  and  $u \leq \xi \leq v$ , then

$$0 < v - u < 2\delta(\xi) \leq 2cv,$$

re-arranging we get  $v/u \leq 1/(1-2c)$ , and finally

$$(v-u)/\sqrt{uv} < 2cv/u \leq 2c/(1-2c).$$

Now choose  $c$  so that  $0 < c < 1/2$  and  $2c/(1-2c) \leq \varepsilon/2$ . In addition, put  $\delta(0) \leq \varepsilon^2/16$ . Then for the given  $\delta$  - fine division the above inequality is less than

$$2\sqrt{\delta(0)} + \frac{2c}{1-2c} \sum_{i=2}^n (\sqrt{x_i} - \sqrt{x_{i-1}}) < \varepsilon$$

For example, when  $0 < \varepsilon \leq 1$  we may choose

$$c = \frac{\varepsilon}{6}. \text{ Hence the function is Henstock-Kurzweil}$$

integrable on  $[0,1]$ .

**CONCLUSION**

From the discussion we get conclusion that:

1.  $\delta$ -fine Henstock-Kurzweil partition  $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$  we write

$$S(f, D) = \sum_{i=1}^n f(\xi_i)(v_i - u_i)$$

where  $D$  be a finite collection of interval-point pairs  $\{([u_i, v_i], \xi_i)\}_{i=1}^n$ , where  $\{([u_i, v_i])\}_{i=1}^n$

are non-overlapping subintervals of  $[a, b]$ .

Let  $\delta(\xi)$  be a positive function on  $[a, b]$ ,

i.e.  $\delta(\xi): [a, b] \rightarrow \mathfrak{R}^+$ .

And if

$$\xi_i \in [u_i, v_i] \subset B(\xi_i, \delta(\xi_i)) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$$

for all  $i = 1, 2, 3, \dots, n$ .

2. A function  $f: [a, b] \rightarrow \mathfrak{R}$  is said to be Henstock-Kurzweil integrable on  $[a, b]$  if there exists a real number  $S_f$  such that for every  $\varepsilon > 0$  there exists  $\delta: [a, b] \rightarrow \mathfrak{R}^+$  such that for every  $\delta$ -fine Henstock-Kurzweil partition  $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$  of  $[a, b]$ , we have

$$|S(f, D) - S_f| < \varepsilon.$$

3. And the fundamental properties of Henstock-Kurzweil integral as follows: value of the Henstock-Kurzweil integral is unique, linearity of the Henstock-Kurzweil integral, Additivity of the Henstock-Kurzweil integral, Cauchy criteria, nonnegativity of Henstock-Kurzweil integral, and primitive function.

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